European Journal of Combinatorics 118 (2024) 103869

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Excedance-type polynomials, gamma-positivity and alternatingly increasing property

Shi-Mei Ma [a], Jun Ma [b], Jean Yeh [c,1], Yeong-Nan Yeh [d,1]

[a] School of Mathematics and Statistics, Northeastern University at Qinhuangdao, 066004, Hebei, China
[b] Department of Mathematics, Shanghai Jiao Tong University, Shanghai, China
[c] Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan
[d] College of Mathematics and Physics, Wenzhou University, Wenzhou 325035, China

A R T I C L E I N F O

Article history:
Received 27 May 2021
Accepted 21 September 2023
Available online xxxx

A B S T R A C T

In this paper, we first give a sufficient condition for a sequence of polynomials to have alternatingly increasing property, and then we present a systematic study of excedance-type polynomials of permutations and derangements, signed or not, colored or not. Let $p \in [0, 1]$ and $q \in [0, 1]$ be two given real numbers. We prove that the cyc q-Eulerian polynomials of permutations are bi-γ-positive, and the fix and cyc (p, q)-Eulerian polynomials of permutations are alternatingly increasing, where fix and cyc are respectively the fixed point and cycle statistics. When $q = 1/2$, we present a combinatorial interpretation of the bi-γ-coefficients of the cyc q-Eulerian polynomials. We then study excedance and flag excedance statistics of signed permutations and colored permutations. We establish relationships between the fix and cyc (p, q)-Eulerian polynomials and some multivariate excedance-type polynomials. In particular, we present a relationship between derangement polynomials of finite Coxeter groups of types A and D. Our results unify and generalize a variety of recent results. Moreover, one can see that the fix and cyc (p, q)-Eulerian polynomials of permutations contain a great deal of information about permutations and colored permutations.

E-mail addresses: shimeimapapers@163.com (S.-M. Ma), majun904@sjtu.edu.cn (J. Ma), chunchenyeh@mail.nknu.edu.tw (J. Yeh), mayeh@math.sinica.edu.tw (Y.-N. Yeh).

1 Corresponding authors.

https://doi.org/10.1016/j.ejc.2023.103869
0195-6698/

1. Introduction

Unimodal polynomials occur naturally in combinatorics, algebra, geometry and analysis. The reader is referred to [3,31,32,37,39,40,42,47,55] for recent progress on this subject. As pointed out by Brenti [12], to prove the unimodality of a polynomial can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools. This paper is motivated by empirical evidence which suggests that some multivariate Eulerian polynomials (with the specialization of some variables) are unimodal with modes in the middle.

Let $f(x)=\sum_{i=0}^{n} f_i x^i$ be a polynomial with real coefficients. We say that $f(x)$ is *unimodal* if

$$f_0 \leqslant f_1 \leqslant \cdots \leqslant f_k \geqslant f_{k+1} \geqslant \cdots \geqslant f_n$$

for some k, where the index k is called the *mode* of $f(x)$. If $f(x)$ is symmetric with the center of symmetry $\lfloor n/2 \rfloor$, i.e., $f_i = f_{n-i}$ for all indices $0 \leqslant i \leqslant n$, then it can be expanded as

$$f(x)=\sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1+x)^{n-2k}.$$

Following Gal [27], the polynomial $f(x)$ is *γ-positive* if $\gamma_k \geqslant 0$ for all $0 \leqslant k \leqslant \lfloor n/2 \rfloor$, and the sequence $\{\gamma_k\}_{k=0}^{\lfloor n/2 \rfloor}$ is called the *γ-vector* of $f(x)$. Clearly, γ-positivity implies symmetry and unimodality. We say that the polynomial $f(x)$ is *spiral* if

$$f_n \leqslant f_0 \leqslant f_{n-1} \leqslant f_1 \leqslant \cdots \leqslant f_{\lfloor n/2 \rfloor}.$$

Following [49, Definition 2.9], the polynomial $f(x)$ is *alternatingly increasing* if

$$f_0 \leqslant f_n \leqslant f_1 \leqslant f_{n-1} \leqslant \cdots \leqslant f_{\lfloor (n+1)/2 \rfloor}.$$

If $f(x)$ is spiral and $\deg f(x) = n$, then $x^n f(1/x)$ is alternatingly increasing, and vice versa. Clearly, spiral property and alternatingly increasing property are stronger than unimodality. The alternatingly increasing property first appeared in the work of Beck–Stapledon [7]. Recently, Beck–Jochemko–McCullough [6] and Solus [53] studied the alternatingly increasing property of several h^*-polynomials as well as some refined Eulerian polynomials.

We now recall an elementary result.

Proposition 1.1 (*[7,10]*). *Let $f(x)$ be a polynomial of degree n. There is a unique symmetric decomposition $f(x) = a(x) + xb(x)$, where*

$$a(x)=\frac{f(x)-x^{n+1}f(1/x)}{1-x},\quad b(x)=\frac{x^n f(1/x)-f(x)}{1-x}. \tag{1}$$

When $f(0)\neq 0$, we have $\deg a(x)=n$ and $\deg b(x)\leqslant n-1$.

We call the ordered pair of polynomials $(a(x), b(x))$ the *symmetric decomposition* of $f(x)$, since $a(x)$ and $b(x)$ are both symmetric polynomials.

Definition 1.2. Let $(a(x), b(x))$ be the *symmetric decomposition* of a polynomial $f(x)$. We say that $f(x)$ is *bi-γ-positive* if $a(x)$ and $b(x)$ are both γ-positive. The γ-coefficients of $a(x)$ and $b(x)$ are called the bi-γ-coefficients of $f(x)$.

Brändén–Solus [10] pointed out that the polynomial $f(x)$ is alternatingly increasing if and only if the pair of polynomials in its symmetric decomposition are both unimodal and have only nonnegative coefficients. Therefore, bi-γ-positivity is stronger than alternatingly increasing property. In Section 2, we discuss the relationship between bi-γ-positivity and alternatingly increasing property. One of the main results is Theorem 2.4, which provides a sufficient condition for a bivariate polynomial to have alternatingly increasing property.

Let $[n]=\{1,2,\ldots,n\}$. Let $\mathcal{S}_n$ be the set of permutations on $[n]$. For $\pi=\pi(1)\cdots\pi(n)\in\mathcal{S}_n$, we say that i is a *descent* (resp. *excedance*, *drop*, *fixed point*) if $\pi(i)>\pi(i+1)$ (resp. $\pi(i)>i$, $\pi(i)<$

i, $\pi(i) = i$). Let $\operatorname{des}(\pi)$, $\operatorname{exc}(\pi)$, $\operatorname{drop}(\pi)$, $\operatorname{fix}(\pi)$ and $\operatorname{cyc}(\pi)$ be the numbers of descents, excedances, drops, fixed points and cycles of π, respectively. Clearly,

$$\operatorname{exc}(\pi) + \operatorname{drop}(\pi) + \operatorname{fix}(\pi) = n.$$

It is well known that descents, excedances and drops are equidistributed over $\mathcal{S}_n$, and their common enumerative polynomial is the classical *Eulerian polynomial* (see [52, A008292]):

$$A_n(x) = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}(\pi)} = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{exc}(\pi)} = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{drop}(\pi)}.$$

The (p, q)-*Eulerian polynomials* $A_n(x, p, q)$ are defined by

$$A_n(x, p, q) = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{exc}(\pi)} p^{\operatorname{fix}(\pi)} q^{\operatorname{cyc}(\pi)}. \tag{2}$$

In this paper, one can see that the (p, q)-Eulerian polynomials $A_n(x, p, q)$ contain a great deal of information about permutations and colored permutations. One can obtain Eulerian polynomials of types A and B, q-derangement polynomials of types A and B, $1/k$-Eulerian polynomials and r-colored Eulerian polynomials from the (p, q)-Eulerian polynomials by special parametrizations.

In Section 3, as an application of Theorem 2.4, we study the alternatingly increasing property of $A_n(x, p, q)$. In Section 4, we first give a combinatorial interpretation of the symmetric decomposition of the $1/k$-Eulerian polynomials $k^n A_n(x, 1, 1/k)$, and then we present a combinatorial interpretation of the bi-γ-coefficients of the following weighted Eulerian polynomials:

$$2^n A_n(x, 1, 1/2) = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{exc}(\pi)} 2^{n-\operatorname{cyc}(\pi)}.$$

In Section 5, we study excedance-type polynomials (in six variables) of signed permutations. In Section 6, we present some results on the enumerative polynomials of signed permutations associated with the flag excedance statistic. In Section 7, we study excedance-type polynomials of colored permutations, including r-colored Eulerian polynomials and three kinds of multivariate colored Eulerian polynomials.

Besides Theorem 2.4, the main results include Theorems 3.4, 3.6, 4.6, 5.2, 5.11, 5.13, 6.2, 6.5, 7.5, 7.8, 7.11 and 7.15. In conclusion, our results unify and generalize some results of Athanasiadis [2,4], Bagno–Garber [5], Brändén–Solus [10], Chen–Tang–Zhao [20], Chow [21,22], Chow–Mansour [23], Foata–Han [25], Han [32], Mongelli [45], Petersen [46], Shin–Zeng [50,51].

2. Bi-gamma-positivity and alternatingly increasing property

If $f(x)$ is γ-positive, then $f(x)$ is also bi-γ-positive but not vice versa. We now provide a connection between γ-positivity and bi-γ-positivity.

Proposition 2.1. *If $f(x)$ is γ-positive and $f(0) = 0$, then $f'(x)$ is bi-γ-positive.*

Proof. Assume that $f(x) = \sum_{k=1}^{\lfloor n/2\rfloor} \gamma_k x^k (1+x)^{n-2k}$, where $\gamma_k \geqslant 0$ for all $1 \leqslant k \leqslant \lfloor n/2\rfloor$. Then we have

$$\begin{aligned} f'(x) &= \sum_{k=1}^{\lfloor n/2\rfloor} k\gamma_k x^{k-1}(1+x)^{n-2k} + \sum_{k=1}^{\lfloor (n-1)/2\rfloor} (n-2k)\gamma_k x^k (1+x)^{n-2k-1} \\ &= \sum_{i=0}^{\lfloor (n-2)/2\rfloor} (i+1)\gamma_{i+1} x^i (1+x)^{n-2i-2} + x \sum_{j=0}^{\lfloor (n-3)/2\rfloor} (n-2j-2)\gamma_{j+1} x^j (1+x)^{n-2j-3}. \end{aligned}$$

Therefore, $f'(x)$ is bi-γ-positive. □

The following simple result will be used repeatedly in our discussion. For completeness, we give a proof of it.

Lemma 2.2. *Let $f(x)=\sum_{i=0}^{n}f_ix^i$ and $g(x)=\sum_{j=0}^{m}g_jx^j$. If $f(x)$ is γ-positive and $g(x)$ is bi-γ-positive, then $f(x)g(x)$ is bi-γ-positive. In particular, the product of two γ-positive polynomials is also γ-positive.*

Proof. Assume that $f(x)=\sum_{k=0}^{\lfloor n/2\rfloor}\gamma_kx^k(1+x)^{n-2k}$ and

$$g(x)=\sum_{i=0}^{\lfloor m/2\rfloor}\xi_ix^i(1+x)^{m-2i}+\sum_{j=1}^{\lfloor (m+1)/2\rfloor}\eta_jx^j(1+x)^{m+1-2j},$$

where γ_k, ξ_i and η_j are all nonnegative numbers. Then $f(x)g(x)$ can be expanded as

$$f(x)g(x)=\sum_{s=0}^{\lfloor (n+m)/2\rfloor}\alpha_sx^s(1+x)^{n+m-2s}+\sum_{t=1}^{\lfloor (n+m+1)/2\rfloor}\beta_tx^t(1+x)^{n+m+1-2t},$$

where $\alpha_s=\sum_{k+i=s}\gamma_k\xi_i\geqslant 0$ and $\beta_t=\sum_{k+j=t}\gamma_k\eta_j\geqslant 0$, as desired. □

We now recall a definition.

Definition 2.3 ([*40*, *Definition 4*]). Let $p(x,y)$ be a bivariate polynomial. Suppose $p(x,y)$ can be expanded as

$$p(x,y)=\sum_{i=0}^{n}y^i\sum_{j=0}^{\lfloor (n-i)/2\rfloor}\mu_{n,i,j}x^j(1+x)^{n-i-2j}. \tag{3}$$

If $\mu_{n,i,j}\geqslant 0$ for all $0\leqslant i\leqslant n$ and $0\leqslant j\leqslant\lfloor (n-i)/2\rfloor$, then we say that $p(x,y)$ is partial γ-positive. The numbers $\mu_{n,i,j}$ are called the partial γ-coefficients of $p(x,y)$.

The partial γ-positive polynomials frequently appear in combinatorics and geometry, see [3,33,34,40] for details. We can now conclude the first main result of this paper.

Theorem 2.4. *Suppose the polynomial $p(x,y)$ has the expression (3) and $\deg p(x,1)=n-1$, where n is a positive integer. If $p(x,y)$ is partial γ-positive, $p(x,1)$ is bi-γ-positive and $0\leqslant y\leqslant 1$ is a given real number, then $p(x,y)$ is alternatingly increasing.*

Proof. For $0\leqslant i\leqslant n$ and $0\leqslant j\leqslant\lfloor (n-i)/2\rfloor$, let

$$\mu_{n,i,j}x^j(1+x)^{n-i-2j}=\sum_{\ell=j}^{n-i-j}S_{n,i,j,\ell}x^\ell.$$

Since the polynomials $\sum_{\ell=j}^{n-i-j}S_{n,i,j,\ell}x^\ell$ are symmetric and unimodal, we have

$$\begin{cases}S_{n,i,j,\ell}=S_{n,i,j,n-i-\ell}, & \text{if } j\leqslant\ell\leqslant n-i-j;\\ S_{n,i,j,\ell}\leqslant S_{n,i,j,\ell+k}, & \text{if } j\leqslant\ell<\ell+k\leqslant\lfloor (n-i)/2\rfloor;\\ S_{n,i,j,\ell}\geqslant S_{n,i,j,\ell+k}, & \text{if } \lfloor (n-i)/2\rfloor\leqslant\ell<\ell+k\leqslant n-i-j.\end{cases} \tag{4}$$

For any $n\geqslant 1$, assume that $p(x,y)=\sum_{\ell=0}^{n-1}p_{n,\ell}(y)x^\ell$ where

$$p_{n,\ell}(y)=\sum_{i=0}^{n}y^i\sum_{j=0}^{\lfloor (n-i)/2\rfloor}S_{n,i,j,\ell}.$$

For $0\leqslant\ell\leqslant\lfloor n/2\rfloor$, we have

$$p_{n,\ell}(y)-p_{n,n-\ell}(y)=\sum_{i=0}^{n}y^i\sum_{j=0}^{\lfloor (n-i)/2\rfloor}\left(S_{n,i,j,\ell}-S_{n,i,j,n-\ell}\right)$$

$$= \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \left(S_{n,i,j,\ell} - S_{n,i,j,\ell-i} \right).$$

By (4), we have $S_{n,i,j,\ell} - S_{n,i,j,\ell-i} \geqslant 0$. Hence $p_{n,\ell}(y) \geqslant p_{n,n-\ell}(y)$ when $y \geqslant 0$.

For $0 \leqslant \ell \leqslant \lfloor n/2 \rfloor - 1$, we have

$$p_{n,n-1-\ell}(y) - p_{n,\ell}(y) = \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \left(S_{n,i,j,n-1-\ell} - S_{n,i,j,\ell} \right)$$
$$= \sum_{i=0}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \left(S_{n,i,j,\ell+1-i} - S_{n,i,j,\ell} \right),$$

which can be rewritten as $p_{n,n-1-\ell}(y) - p_{n,\ell}(y) = P_{n,\ell} - Q_{n,\ell}(y)$, where

$$P_{n,\ell} = \sum_{j=0}^{\lfloor n/2 \rfloor} \left(S_{n,0,j,\ell+1} - S_{n,0,j,\ell} \right), \quad Q_{n,\ell}(y) = \sum_{i=1}^{n} y^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \left(S_{n,i,j,\ell} - S_{n,i,j,\ell+1-i} \right).$$

It follows from (4) that $S_{n,0,j,\ell+1} - S_{n,0,j,\ell} \geqslant 0$ and $S_{n,i,j,\ell} - S_{n,i,j,\ell+1-i} \geqslant 0$ for $i \geqslant 1$. Since $p(x,1) = \sum_{\ell=0}^{n-1} p_{n,\ell}(1)x^\ell$ is bi-γ-positive, the polynomial $p(x,1)$ is alternatingly increasing, which implies that

$$p_{n,n-1-\ell}(1) - p_{n,\ell}(1) = P_{n,\ell} - Q_{n,\ell}(1) \geqslant 0. \tag{5}$$

Therefore, if $0 \leqslant y \leqslant 1$, then $P_{n,\ell} - Q_{n,\ell}(y) \geqslant P_{n,\ell} - Q_{n,\ell}(1) \geqslant 0$. In conclusion, when $0 \leqslant y \leqslant 1$, we get $p_{n,\ell}(y) \geqslant p_{n,n-\ell}(y)$ for $0 \leqslant \ell \leqslant \lfloor n/2 \rfloor$ and $p_{n,n-1-\ell}(y) \geqslant p_{n,\ell}(y)$ for $0 \leqslant \ell \leqslant \lfloor n/2 \rfloor - 1$. This completes the proof. □

In the next Section, we shall present an application of Theorem 2.4.

3. Excedance-type polynomials of permutations

3.1. Preliminary

The cardinality of a set A will be denoted by $\#A$. Let $\pi \in \mathcal{S}_n$. Recall that

$$\operatorname{des}(\pi) = \#\{i \in [n-1] : \pi(i) > \pi(i+1)\}.$$

An index $i \in [n]$ is called a *double descent* of π if $\pi(i-1) > \pi(i) > \pi(i+1)$, where $\pi(0) = \pi(n+1) = 0$. Foata–Schützenberger [26] found the following notable result.

Proposition 3.1 (*[26]*). *For $n \geqslant 1$, one has*

$$A_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i} x^i (1+x)^{n-1-2i},$$

where $\gamma_{n,i}$ is the number of permutations $\pi \in \mathcal{S}_n$ which have no double descents and $\operatorname{des}(\pi) = i$.

An element $\pi \in \mathcal{S}_n$ is called a *derangement* if $\operatorname{fix}(\pi) = 0$. Let $\mathcal{D}_n$ be the set of all derangements in $\mathcal{S}_n$. The *derangement polynomials* (of type A) are defined by

$$d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\operatorname{exc}(\pi)}.$$

The exponential generating function of $d_n(x)$ is given as follows (see [11, Proposition 6]):

$$d(x;z)=\sum_{n=0}^{\infty}d_n(x)\frac{z^n}{n!}=\frac{1-x}{e^{xz}-xe^{z}}. \tag{6}$$

The reader is referred to [52, A046739] for some references on derangement polynomials.

Let $\mathrm{cda}(\pi)=\#\{i:\pi^{-1}(i)<i<\pi(i)\}$ be the number of *cycle double ascents* of π. Using the theory of continued fractions, Shin–Zeng [50, Theorem 11] obtained the following result.

Proposition 3.2 (*[50]*). *Let $\mathcal{D}_{n,k}=\{\pi\in\mathcal{S}_n:\mathrm{fix}(\pi)=0,\ \mathrm{cda}(\pi)=0,\ \mathrm{exc}(\pi)=k\}$. Then*

$$d_n(x,q)=\sum_{\pi\in\mathcal{D}_n}x^{\mathrm{exc}(\pi)}q^{\mathrm{cyc}(\pi)}=\sum_{k=1}^{\lfloor n/2\rfloor}\sum_{\pi\in\mathcal{D}_{n,k}}q^{\mathrm{cyc}(\pi)}x^k(1+x)^{n-2k}. \tag{7}$$

In the past decades, various refinements and generalizations of Propositions 3.1 and 3.2 have been extensively studied. The reader is referred to [3,28,33,40,50,55] for more details.

Let $\pm[n]=[n]\cup\{\bar{1},\ldots,\bar{n}\}$, where $\bar{i}=-i$. Let $\mathcal{S}_n^B$ be the *hyperoctahedral group* of rank n. Elements of $\mathcal{S}_n^B$ are permutations of $\pm[n]$ with the property that $\sigma(\bar{i})=-\sigma(i)$ for all $i\in[n]$. Let $\sigma=\sigma(1)\sigma(2)\cdots\sigma(n)\in\mathcal{S}_n^B$. An *excedance* (resp. *fixed point*) of σ is an index $i\in[n]$ such that $\sigma(|\sigma(i)|)>\sigma(i)$ (resp. $\sigma(i)=i$). Let $\mathrm{exc}(\sigma)$ (resp. $\mathrm{neg}(\sigma)$, $\mathrm{fix}(\sigma)$ and $\mathrm{cyc}(\sigma)$) denote the number of excedances (resp. negative elements, fixed points and cycles) of σ. Let $\mathcal{D}_n^B=\{\sigma\in\mathcal{S}_n^B:\mathrm{fix}(\sigma)=0\}$ be the set of all derangements in $\mathcal{S}_n^B$. The *type B derangement polynomials* $d_n^B(x)$ are defined by

$$d_n^B(x)=\sum_{\sigma\in\mathcal{D}_n^B}x^{\mathrm{exc}(\sigma)}.$$

The polynomials $d_n^B(x)$ have been studied by Chen–Tang–Zhao [20], Chow [22] and Shin–Zeng [51]. According to [22, Theorem 3.2], the exponential generating function of $d_n^B(x)$ is given as follows:

$$\sum_{n=0}^{\infty}d_n^B(x)\frac{z^n}{n!}=\frac{(1-x)\mathrm{e}^{z}}{\mathrm{e}^{2xz}-x\mathrm{e}^{2z}}. \tag{8}$$

Chen-Tang-Zhao [20, Theorem 4.6] proved the following remarkable result.

Proposition 3.3 (*[20]*). *For $n\geqslant 1$, the polynomials $x^n d_n^B(1/x)$ are spiral. Equivalently, the polynomials $d_n^B(x)$ are alternatingly increasing.*

For the (p,q)-Eulerian polynomials $A_n(x,p,q)$ defined by (2), we set

$$A_n(x,q)=A_n(x,1,q)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{exc}(\pi)}q^{\mathrm{cyc}(\pi)}.$$

Brenti [14] showed that some of the crucial properties of Eulerian polynomials have nice q-analogues for the q-Eulerian polynomials $A_n(x,q)$. Following [14, Proposition 7.2], the polynomials $A_n(x,q)$ satisfy the recurrence relation

$$A_{n+1}(x,q)=(nx+q)A_n(x,q)+x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n(x,q),\ A_0(x,q)=1. \tag{9}$$

According to [14, Proposition 7.3], we have

$$\sum_{n=0}^{\infty}A_n(x,q)\frac{z^n}{n!}=\left(\frac{(1-x)\mathrm{e}^{z}}{\mathrm{e}^{xz}-x\mathrm{e}^{z}}\right)^q.$$

Using the *exponential formula*, Ksavrelof and Zeng [36] found that

$$\sum_{n=0}^{\infty}A_n(x,p,q)\frac{z^n}{n!}=\left(\frac{(1-x)\mathrm{e}^{pz}}{\mathrm{e}^{xz}-x\mathrm{e}^{z}}\right)^q. \tag{10}$$

Below are the polynomials $A_n(x,p,q)$ for $n \leqslant 4$:

$$A_1(x,p,q) = pq,\ A_2(x,p,q) = p^2q^2 + qx,\ A_3(x,p,q) = p^3q^3 + (q+3pq^2)x + qx^2,$$
$$A_4(x,p,q) = p^4q^4 + (q+4pq^2+6p^2q^3)x + (4q+3q^2+4pq^2)x^2 + qx^3.$$

3.2. Main results

Note that $A_n(x,1,1) = A_n(x)$ and $A_n(x,0,q) = d_n(x,q)$. We can now present the second main result of this paper, which unifies Propositions 3.1 and 3.2.

Theorem 3.4.

(i) For any $n \geqslant 1$, we have

$$A_n(x,p,q) = \sum_{i=0}^{n} p^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j}(q) x^j (1+x)^{n-i-2j}. \tag{11}$$

(ii) Let $\mathcal{S}_{n,i,j} = \{\pi \in \mathcal{S}_n : \mathrm{cda}(\pi) = 0,\ \mathrm{fix}(\pi) = i,\ \mathrm{exc}(\pi) = j\}$. Then

$$\gamma_{n,i,j}(q) = \sum_{\pi \in \mathcal{S}_{n,i,j}} q^{\mathrm{cyc}(\pi)}. \tag{12}$$

Therefore, the (p,q)-Eulerian polynomial $A_n(x,p,q)$ is partial γ-positive if $q \geqslant 0$ is a given real number.

(iii) Let

$$\gamma = \gamma(x,p,q;z) = 1 + \sum_{n=1}^{\infty} \sum_{i=0}^{n} p^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j}(q) x^j \frac{z^n}{n!}.$$

Then we have

$$\gamma(x,p,q;z) = e^{z\left(p-\frac{1}{2}\right)q} \left(\frac{\sqrt{1-4x}}{\sqrt{1-4x}\cosh\left(\frac{z}{2}\sqrt{1-4x}\right) - \sinh\left(\frac{z}{2}\sqrt{1-4x}\right)} \right)^q. \tag{13}$$

A *left peak* of $\pi \in \mathcal{S}_n$ is an index $i \in [n-1]$ such that $\pi(i-1) < \pi(i) > \pi(i+1)$, where $\pi(0) = 0$. Denote by $\mathrm{lpk}(\pi)$ the number of left peaks in π. Let $Q(n,i) = \#\{\pi \in \mathcal{S}_n : \mathrm{lpk}(\pi) = i\}$, and let $Q_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} Q(n,i)x^i$. Gessel [52, A008971] obtained the following exponential generating function:

$$Q(x;z) = 1 + \sum_{n=1}^{\infty} Q_n(x)\frac{z^n}{n!} = \frac{\sqrt{1-x}}{\sqrt{1-x}\cosh(z\sqrt{1-x}) - \sinh(z\sqrt{1-x})}. \tag{14}$$

Comparing (13) with (14) leads to

$$Q(x;z) = \gamma\left(\frac{x}{4}, \frac{1}{2}, 1; 2z\right). \tag{15}$$

Let $\mathcal{C}_n$ be the set of permutations in $\mathcal{S}_n$ with no cycle double ascents. Note that

$$\sum_{\pi \in \mathcal{C}_n} x^{\mathrm{exc}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{cyc}(\pi)} = \sum_{i=0}^{n} \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j}(q) p^i x^j.$$

An equivalent result to (15) is given as follows.

Corollary 3.5. *We have*

$$\sum_{\pi \in \mathcal{S}_n} x^{\mathrm{lpk}(\pi)} = \sum_{\pi \in \mathcal{C}_n} 2^{n-\mathrm{fix}(\pi)-2\mathrm{exc}(\pi)} x^{\mathrm{exc}(\pi)}.$$

Comparing (8) with (10), we get $2^n A_n(x, 1/2, 1) = d_n^B(x)$. From Proposition 3.3, we see that $A_n(x, 1/2, 1)$ are alternatingly increasing. The third main result of this paper is given as follows.

Theorem 3.6. *Let $p \in [0, 1]$ and $q \in [0, 1]$ be two given real numbers, i.e., $0 \leqslant p \leqslant 1$ and $0 \leqslant q \leqslant 1$. Then we have the following two results:*

(i) the polynomials $A_n(x, q) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} q^{\mathrm{cyc}(\pi)}$ are bi-γ-positive;
(ii) the polynomials $A_n(x, p, q) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{cyc}(\pi)}$ are alternatingly increasing.

In Sections 5–7, we collect several applications of Theorem 3.6.

3.3. Proof of Theorem 3.4

Following Chen [16], a *context-free grammar* G over an alphabet V is defined as a set of substitution rules replacing a letter in V by a formal function over V. As usual, the formal function may be a polynomial or a Laurent polynomial. The formal derivative D_G with respect to G satisfies the derivation rules:

$$D_G(u + v) = D_G(u) + D_G(v), \ \ D_G(uv) = D_G(u)v + uD_G(v).$$

So the Leibniz rule holds:

$$D_G^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_G^k(u) D_G^{n-k}(v).$$

Example 3.7 ([16]). Let $G = \{x \to xy, y \to y\}$. Then

$$D_G^n(x) = x \sum_{k=0}^{n} \left\{ {n \atop k} \right\} y^k,$$

where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second kind.

In recent years, context-free grammars have been used to study permutations [17,19,40], increasing trees [17,19], Stirling permutations [18,40,44] and perfect matchings [38]. According to [17], an advantage of the grammatical description of a combinatorial sequence is that a recursion of its generating function can be provided by attaching a labeling of the combinatorial object in accordance with the replacement rules of the grammar.

The following two definitions will be used repeatedly in our discussion.

Definition 3.8 ([17]). A grammatical labeling is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar.

Definition 3.9 ([40]). A change of grammar is a substitution method in which the original grammar is replaced with functions of other grammars.

The method of change of grammar has proved to be useful in handling combinatorial expansions of descent-type polynomials, see [40,41] for details. In this paper, we use the change of grammar technique to establish combinatorial expansions of excedance-type polynomials.

Let $A_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$, where $\left\langle {n \atop k} \right\rangle$ are called *Eulerian numbers* (see [52, A008292]). There is a grammatical interpretation for Eulerian numbers.

Proposition 3.10 ([24, Section 2.1]). *If $V = \{x, y\}$ and $G = \{x \to xy, y \to xy\}$, then*

$$D_G^n(x) = x \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k y^{n-k} \quad \text{for } n \geqslant 1.$$

For $\pi \in \mathcal{S}_n$, recall that $\operatorname{exc}(\pi)+\operatorname{drop}(\pi)+\operatorname{fix}(\pi)=n$. Now we present a simple variation of $A_n(x,p,q)$, which will be used repeatedly in this paper.

Proposition 3.11. *We have*

$$\sum_{\pi\in\mathcal{S}_n} x^{\operatorname{exc}(\pi)}y^{\operatorname{drop}(\pi)}p^{\operatorname{fix}(\pi)}q^{\operatorname{cyc}(\pi)} = y^n\sum_{\pi\in\mathcal{S}_n}\left(\frac{x}{y}\right)^{\operatorname{exc}(\pi)}\left(\frac{p}{y}\right)^{\operatorname{fix}(\pi)}q^{\operatorname{cyc}(\pi)} = y^nA_n\left(\frac{x}{y},\frac{p}{y},q\right).$$

In the following discussion, we always write permutation, signed or not, by its standard cycle form, in which each cycle has its smallest (in absolute value) element first and the cycles are written in increasing order of the absolute value of their first elements. As an extension of Proposition 3.10, we present a fundamental lemma.

Lemma 3.12. *Let $V=\{I,p,q,x,y\}$ and $G=\{I\to Ipq, p\to xy, x\to xy, y\to xy, q\to 0\}$. Then*

$$D_G^n(I)=I\sum_{\pi\in\mathcal{S}_n} x^{\operatorname{exc}(\pi)}y^{\operatorname{drop}(\pi)}p^{\operatorname{fix}(\pi)}q^{\operatorname{cyc}(\pi)}.$$

Proof. Let $\pi\in\mathcal{S}_n$. We introduce a grammatical labeling of π:

(L_1) if i is an excedance, then put a superscript label x right after i;
(L_2) if i is a drop, then put a superscript label y right after i;
(L_3) if i is a fixed point, then put a superscript label p right after i;
(L_4) put a superscript label I right after π and put a subscript label q right after each cycle.

With this labeling, the weight of π is defined as the product of its labels, that is,

$$w(\pi)=Ix^{\operatorname{exc}(\pi)}y^{\operatorname{drop}(\pi)}p^{\operatorname{fix}(\pi)}q^{\operatorname{cyc}(\pi)}.$$

For example, let $\pi=(1,4,3)(2,6)(5)$. The grammatical labeling of π is given below

$$(1^x4^y3^y)_q(2^x6^y)_q(5^p)_q^I.$$

If we insert 7 after 5, the resulting permutation is $(1^x4^y3^y)_q(2^x6^y)_q(5^x7^y)_q^I$. If the inserted 7 forms a new cycle, the resulting permutation is $(1^x4^y3^y)_q(2^x6^y)_q(5^p)_q(7^p)_q^I$. If we insert 7 after 1, 4, 3, 2 or 6, then we respectively get

$$(1^x7^y4^y3^y)_q(2^x6^y)_q(5^p)_q^I,$$

$$(1^x4^x7^y3^y)_q(2^x6^y)_q(5^p)_q^I,$$

$$(1^x4^y3^x7^y)_q(2^x6^y)_q(5^p)_q^I,$$

$$(1^x4^y3^y)_q(2^x7^y6^y)_q(5^p)_q^I,$$

$$(1^x4^y3^y)_q(2^x6^x7^y)_q(5^p)_q^I.$$

In each case, the insertion of 7 corresponds to one substitution rule in G. When $n=1,2$, we have $\mathcal{S}_1=\{(1^p)_q^I\}$ and $\mathcal{S}_2=\{(1^p)_q(2^p)_q^I,\ (1^x2^y)_q^I\}$. Consider $\pi\in\mathcal{S}_{n-1}$, where $n\geqslant 3$. When we insert the entry n into π, we only need to distinguish among four distinct cases:

(c_1) $\pi^I\to\pi(n^p)_q^I$;
(c_2) $\cdots(i^p)_q\cdots\to\cdots(i^xn^y)_q\cdots$;
(c_3) $\cdots(\cdots i^xj\cdots)_q\cdots\to\cdots(\cdots i^xn^yj\cdots)_q\cdots$;
(c_4) $\cdots(\cdots i^yj\cdots)_q\cdots\to\cdots(\cdots i^xn^yj\cdots)_q\cdots$.

Therefore, the action of the formal derivative D_G on the set of weighted permutations in $\mathcal{S}_{n-1}$ gives the set of weighted permutations in $\mathcal{S}_n$. This yields the desired result. $\square$

It should be noted that the grammar given in Lemma 3.12 can be simplified as follows:

$$G = \{I \to Ipq, p \to xy, x \to xy, y \to xy\}, \tag{16}$$

where q can be seen as a given parameter and $D_G(q) = 0$.

A proof of the combinatorial expansion (11). By Lemma 3.12, we obtain

$$D_G^n(I) \mid_{I=y=1} = A_n(x, p, q). \tag{17}$$

Consider a change of the grammar given by (16). Let $u = xy$ and $v = x + y$. Then

$$D_G(I) = Ipq,\ D_G(p) = u,\ D_G(u) = uv,\ D_G(v) = 2u.$$

Let $G_2 = \{I \to Ipq, p \to u, u \to uv, v \to 2u\}$. Note that

$$D_{G_2}(I) = Ipq, D_{G_2}^2(I) = I(p^2q^2 + qu), D_{G_2}^3(I) = I(p^3q^3 + 3pq^2u + quv).$$

For $n \geqslant 1$, assume that

$$D_{G_2}^n(I) = I \sum_{i=0}^{n} p^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j}(q) u^j v^{n-i-2j}. \tag{18}$$

We now show that (18) holds for $n + 1$. Since $D_{G_2}^{n+1}(I) = D_{G_2}\left(D_{G_2}^n(I)\right)$, we have

$$\begin{aligned} D_{G_2}^{n+1}(I) &= D_{G_2}\left(I \sum_{i,j} \gamma_{n,i,j}(q) p^i u^j v^{n-i-2j}\right) \\ &= I \sum_{i,j} \gamma_{n,i,j}(q)\left(qp^{i+1}u^j v^{n-i-2j} + ip^{i-1}u^{j+1}v^{n-i-2j} + jp^i u^j v^{n+1-i-2j}\right) + \\ & I \sum_{i,j} \gamma_{n,i,j}(q)(2n - 2i - 4j)p^i u^{j+1} v^{n-1-i-2j}. \end{aligned}$$

Taking the coefficients of $Ip^iu^jv^{n+1-i-2j}$ on the right side of the above expression, we obtain

$$q\gamma_{n,i-1,j}(q) + (i + 1)\gamma_{n,i+1,j-1}(q) + j\gamma_{n,i,j}(q) + (2n - 2i - 4j + 4)\gamma_{n,i,j-1}(q).$$

Hence $D_{G_2}^{n+1}(I)$ can be written as follows:

$$D_{G_2}^{n+1}(I) = I \sum_{i=0}^{n+1} p^i \sum_{j=0}^{\lfloor (n+1-i)/2 \rfloor} \gamma_{n+1,i,j}(q) u^j v^{n+1-i-2j},$$

where

$$\gamma_{n+1,i,j}(q) = q\gamma_{n,i-1,j}(q) + (i + 1)\gamma_{n,i+1,j-1}(q) + j\gamma_{n,i,j}(q) + (2n - 2i - 4j + 4)\gamma_{n,i,j-1}(q), \tag{19}$$

with the initial conditions $\gamma_{1,1,0}(q) = q$ and $\gamma_{1,i,j}(q) = 0$ for $(i, j) \neq (1, 0)$. Hence (18) holds for $n+1$. From (19), we see that if $q \geqslant 0$, then $\gamma_{n,i,j}(q) \geqslant 0$. Moreover, upon substituting $u = xy$ and $v = x+y$ in (18), we get the following expansion:

$$D_G^n(I) = I \sum_{i=0}^{n} p^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j}(q)(xy)^j (x + y)^{n-i-2j}.$$

Comparing this with (17), we obtain

$$A_n(x, p, q) = \sum_{i=0}^{n} p^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j}(q) x^j (1 + x)^{n-i-2j}. \quad \square$$

Let $(c_1, c_2, \ldots, c_i)$ be a cycle of π. Then $c_1 = \min\{c_1, \ldots, c_i\}$. Set $c_{i+1} = c_1$. Then c_j is called

- a *cycle double ascent* in the cycle if $c_{j-1} < c_j < c_{j+1}$, where $2 \leqslant j \leqslant i-1$;
- a *cycle double descent* in the cycle if $c_{j-1} > c_j > c_{j+1}$, where $2 < j \leqslant i$;
- a *cycle peak* in the cycle if $c_{j-1} < c_j > c_{j+1}$, where $2 \leqslant j \leqslant i$;
- a *cycle valley* in the cycle if $c_{j-1} > c_j < c_{j+1}$, where $2 < j \leqslant i-1$.

Let $\mathrm{cda}(\pi)$ (resp. $\mathrm{cdd}(\pi)$, $\mathrm{cpk}(\pi)$, $\mathrm{cval}(\pi)$) be the numbers of cycle double ascents (resp. cycle double descents, cycle peaks, cycle valleys) of π.

We define an action φ_x on $\mathcal{S}_n$ as follows. Let $c = (c_1, c_2, \ldots, c_i)$ be a cycle of $\pi \in \mathcal{S}_n$ with at least two elements. Consider the following three cases:

- if c_k is a cycle double ascent in c, then $\varphi_{c_k}(\pi)$ is obtained by deleting c_k and then inserting c_k between c_j and c_{j+1}, where j is the smallest index satisfying $k < j \leqslant i$ and $c_j > c_k > c_{j+1}$;
- if c_k is a cycle double descent in c, then $\varphi_{c_k}(\pi)$ is obtained by deleting c_k and then inserting c_k between c_j and c_{j+1}, where j is the largest index satisfying $1 \leqslant j < k$ and $c_j < c_k < c_{j+1}$;
- if c_k is neither a cycle double ascent nor a cycle double descent in c, then c_k is a cycle peak or a cycle valley. In this case, we let $\varphi_{c_k}(\pi) = \pi$.

Following [9], we define a *modified Foata–Strehl group action* φ'_x on $\mathcal{S}_n$ by

$$\varphi'_x(\pi) = \begin{cases} \varphi_x(\pi), & \text{if } x \text{ is a cycle double ascent or a cycle double descent;} \\ \pi, & \text{if } x \text{ is a cycle peak or a cycle valley.} \end{cases}$$

Define

$$\mathrm{CDD}(\pi) = \{x \mid x \text{ is a cycle double descent of } \pi\},$$

$$\mathcal{S}^1_{n,i,j,k} = \{\pi \in \mathcal{S}_n : \mathrm{cda}(\pi) = 0,\ \mathrm{fix}(\pi) = i,\ \mathrm{exc}(\pi) = j,\ \mathrm{cyc}(\pi) = k\},$$

$$\mathcal{S}^2_{n,i,j,k} = \{\pi \in \mathcal{S}_n : \mathrm{cda}(\pi) = 1,\ \mathrm{fix}(\pi) = i,\ \mathrm{exc}(\pi) = j,\ \mathrm{cyc}(\pi) = k\}.$$

For $\pi \in \mathcal{S}^1_{n,i,j,k}$ and $x \in \mathrm{CDD}(\pi)$, it should be noted that $\mathrm{exc}(\pi)$ equals the number of cycle peaks of π, $\varphi'_x(\pi) \in \mathcal{S}^2_{n,i,j+1,k}$ and x is the unique cycle double ascent of $\varphi'_x(\pi)$. Conversely, for $\widetilde{\pi} \in \mathcal{S}^2_{n,i,j+1,k}$, let x be the unique cycle double ascent of $\widetilde{\pi}$. Note that $\varphi'_x(\widetilde{\pi}) \in \mathcal{S}^1_{n,i,j,k}$ and x becomes a cycle double descent in $\varphi'_x(\widetilde{\pi})$. This implies that

$$|\mathcal{S}^2_{n,i,j+1,k}| = (n-i-2j)|\mathcal{S}^1_{n,i,j,k}|, \tag{20}$$

where $n-i-2j$ is the number of cycle double descents of permutations in $\mathcal{S}^1_{n,i,j,k}$.

Example 3.13. Let $\pi = (1, 10, 6, 5, 7, 3, 2, 8)(4, 9) \in \mathcal{S}^1_{10,0,4,2}$. We have $\mathrm{CDD}(\pi) = \{3, 6\}$. Then

$$\varphi'_3(\pi) = (1, 3, 10, 6, 5, 7, 2, 8)(4, 9),\ \ \varphi'_6(\pi) = (1, 6, 10, 5, 7, 3, 2, 8)(4, 9),$$

and $\varphi'_3(\pi),\ \varphi'_6(\pi) \in \mathcal{S}^2_{10,0,5,2}$.

A proof of (12) by combining modified Foata–Strehl group action. In order to get a permutation enumerated by $\gamma_{n+1,i,j}(q)$ by inserting the entry $n+1$ into a permutation $\pi \in \mathcal{S}_n$, then either $\mathrm{cda}(\pi) = 0$ or $\mathrm{cda}(\pi) = 1$. When $\mathrm{cda}(\pi) = 0$, we distinguish among four distinct cases:

- (c_1) if $\pi \in \mathcal{S}_{n,i-1,j}$, then we should append $(n+1)$ to π as a new cycle. This accounts for the term $q\gamma_{n,i-1,j}(q)$;
- (c_2) if $\pi \in \mathcal{S}_{n,i+1,j-1}$, then we should insert the entry $n+1$ right after a fixed point. This accounts for the term $(1+i)\gamma_{n,i+1,j-1}(q)$;
- (c_3) if $\pi \in \mathcal{S}_{n,i,j}$, then we should insert the entry $n+1$ right after an excedance. This accounts for the term $j\gamma_{n,i,j}(q)$;

(c_4) if $\pi \in \mathcal{S}^1_{n,i,j-1,k}$, then there are $n-i-2(j-1)$ positions could be inserted the entry $n+1$, since we cannot insert the entry $n+1$ immediately before or right after each cycle peak, and we cannot insert the entry $n+1$ right after a fixed point. This accounts for the term $(n-i-2j+2)\gamma_{n,i,j-1}(q)$.

When $\mathrm{cda}(\pi)=1$ and $\pi \in \mathcal{S}^2_{n,i,j,k}$, let x the unique cycle double ascent of π. We should insert the entry $n+1$ into π immediately before x. Using (20), we get the additional term

$$(n-i-2j+2)\gamma_{n,i,j-1}(q).$$

Thus (19) holds. □

A proof of (13). Define

$$\gamma_n(x,p,q)=\sum_{i=0}^{n}\sum_{j=0}^{\lfloor (n-i)/2\rfloor}\gamma_{n,i,j}(q)p^ix^j,\ \ \gamma := \gamma(x,p,q;z)=\sum_{n=0}^{\infty}\gamma_n(x,p,q)\frac{z^n}{n!}.$$

Multiplying both sides of (19) by p^ix^j and summing over all i and j, we get

$$\gamma_{n+1}(x,p,q)=(pq+2nx)\gamma_n(x,p,q)+x(1-2p)\frac{\partial \gamma_n(x,p,q)}{\partial p}+x(1-4x)\frac{\partial \gamma_n(x,p,q)}{\partial x}.$$

Multiplying both sides of the above recurrence by $\frac{z^n}{n!}$ and summing over all $n\geqslant 0$, we get

$$\frac{\partial \gamma}{\partial z}=pq\gamma+x(1-2p)\frac{\partial \gamma}{\partial p}+x(1-4x)\frac{\partial \gamma}{\partial x}+2xz\frac{\partial \gamma}{\partial z}. \tag{21}$$

One can directly check that the exponential generating function

$$\widehat{\gamma}(x,p,q;z)=e^{z\left(p-\frac{1}{2}\right)q}\left(\frac{\sqrt{1-4x}}{\sqrt{1-4x}\cosh\left(\frac{z}{2}\sqrt{1-4x}\right)-\sinh\left(\frac{z}{2}\sqrt{1-4x}\right)}\right)^q$$

satisfies (21). Also, this exponential generating function gives $\widehat{\gamma}(x,p,q;0)=\widehat{\gamma}(x,p,0;z)=1$. Hence $\widehat{\gamma}(x,p,q;z)=\gamma(x,p,q;z)$. This completes the proof. □

3.4. *Proof of Theorem 3.6*

Let k be a fixed positive integer. The $1/k$-*Eulerian polynomials* $A_n^{(k)}(x)$ are defined by

$$\sum_{n=0}^{\infty}A_n^{(k)}(x)\frac{z^n}{n!}=\left(\frac{1-x}{e^{kz(x-1)}-x}\right)^{\frac{1}{k}}. \tag{22}$$

Let $e=(e_1,e_2,\ldots,e_n)\in\mathbb{Z}^n$. Let $I_{n,k}=\{e|0\leqslant e_i\leqslant (i-1)k\}$ be the set of n-dimensional k-*inversion sequences*. The number of *ascents* of e is defined by

$$\mathrm{asc}(e)=\#\left\{i:1\leqslant i\leqslant n-1 \mid \frac{e_i}{(i-1)k+1}<\frac{e_{i+1}}{ik+1}\right\}.$$

Savage–Viswanathan [48] showed that

$$A_n^{(k)}(x)=\sum_{e\in I_{n,k}}x^{\mathrm{asc}(e)}.$$

In other words, the polynomial $A_n^{(k)}(x)$ is the s-*Eulerian polynomial* of the s-*inversion sequence* $(1,k+1,2k+1,\ldots,(n-1)k+1)$. Comparing (10) with (22), we see that

$$A_n^{(k)}(x)=k^nA_n(x,1,1/k)=k^nA_n(x,1/k)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{exc}(\pi)}k^{n-\mathrm{cyc}(\pi)}. \tag{23}$$

Recently, a bijective proof of (23) was provided in [15].

Let $A_n^{(k)}(x)=\sum_{j=0}^{n-1}A_{n,j;k}x^j$ for $n\geqslant 1$. Below are the polynomials $A_n^{(k)}(x)$ for $n\leqslant 3$:

$$A_1^{(k)}(x)=1,\ A_2^{(k)}(x)=1+kx,\ A_3^{(k)}(x)=1+3kx+k^2x+k^2x^2.$$

Comparing (23) with (9), we get

$$A_{n+1}^{(k)}(x)=(1+nkx)A_n^{(k)}(x)+kx(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_n^{(k)}(x). \tag{24}$$

Extracting the coefficients of x^j in both sides of (24), one can get

$$A_{n+1,j;k}=(1+kj)A_{n,j;k}+k(n-j+1)A_{n,j-1;k}, \tag{25}$$

with the initial conditions $A_{1,0;k}=1$ and $A_{1,i;k}=0$ for $i\neq 0$ (see [43, p. 1470]). Now we give a grammatical description of the coefficients $A_{n,j;k}$.

Lemma 3.14. *If $G_0=\{I\rightarrow Iy,\ x\rightarrow kxy,\ y\rightarrow kxy\}$, then we have*

$$D_{G_0}^n(I)=I\sum_{j=0}^{n-1}A_{n,j;k}x^jy^{n-j}\ \text{ for } n\geqslant 1. \tag{26}$$

Proof. Note that $D_{G_0}(I)=Iy, D_{G_0}^2(I)=I(y^2+kxy)$. Hence the result holds for $n=1,2$. Assume that (26) holds for some n, where $n\geqslant 2$. Note that

$$D_{G_0}^{n+1}(I)=D_{G_0}\left(D_{G_0}^n(I)\right)=I\sum_j A_{n,j;k}\left(x^jy^{n-j+1}+kjx^jy^{n-j+1}+k(n-j)x^{j+1}y^{n-j}\right).$$

Extracting the coefficients of x^jy^{n-j+1} in the right side of the above expression, we get

$$(1+kj)A_{n,j;k}+k(n-j+1)A_{n,j-1;k}.$$

Comparing the above expression of coefficients and (25), then $D_{G_0}^{n+1}(I)$ can be written as

$$D_{G_0}^{n+1}(I)=I\sum_{j=0}^{n}A_{n+1,j;k}x^jy^{n+1-j}.$$

Therefore, the result holds for $n+1$. The proof follows by induction. □

A proof of Theorem 3.6. Let $q\in[0,1]$ be a given real number and let k be a given positive integer. From (23), we see that $A_n^{(k)}(x)=k^nA_n(x,1/k)$. To prove the bi-$\gamma$-positivity of $A_n(x,q)$, it suffices to prove that the polynomials $A_n^{(k)}(x)$ are bi-γ-positive. Consider a change of the grammar given in Lemma 3.14. Note that

$$D_{G_0}(I)=Iy,\ D_{G_0}(Iy)=Iy(x+y)+(k-1)Ixy,$$

$$D_{G_0}(x+y)=2kxy,\ D_{G_0}(xy)=kxy(x+y).$$

Set $J=Iy$, $u=x+y$ and $v=xy$. Then

$$D_{G_0}(I)=J,\ D_{G_0}(J)=Ju+(k-1)Iv,\ D_{G_0}(u)=2kv,\ D_{G_0}(v)=kuv.$$

Let $G_1=\{I\rightarrow J, J\rightarrow Ju+(k-1)Iv, u\rightarrow 2kv, v\rightarrow kuv\}$. By induction, it is routine to verify that there are nonnegative integers such that

$$D_{G_1}^n(I)=J\sum_{i=0}^{\lfloor(n-1)/2\rfloor}A_{n,i;k}^{+}v^iu^{n-1-2i}+Iv\sum_{i=0}^{\lfloor(n-2)/2\rfloor}A_{n,i;k}^{-}v^iu^{n-2-2i}. \tag{27}$$

In particular, $D_{G_1}(I)=J, D_{G_1}^2(I)=Ju+(k-1)Iv$. Note that

$$D_{G_1}^{n+1}(I)=(Ju+(k-1)Iv)\sum_i A_{n,i;k}^{+}v^iu^{n-1-2i}+Jv\sum_i A_{n,i;k}^{-}v^iu^{n-2-2i}+$$

$$J\sum_i A^+_{n,i;k}\left(kiv^iu^{n-2i}+2k(n-1-2i)v^{i+1}u^{n-2-2i}\right)+$$
$$I\sum_i A^-_{n,i;k}\left(k(i+1)v^{i+1}u^{n-1-2i}+2k(n-2-2i)v^{i+2}u^{n-3-2i}\right).$$

Taking coefficients of Jv^iu^{n-2i} and $Iv^{i+1}u^{n-1-2i}$ on both sides yields the recurrence system

$$\begin{cases}A^+_{n+1,i;k}=(1+ki)A^+_{n,i;k}+2k(n-2i+1)A^+_{n,i-1;k}+A^-_{n,i-1;k},\\ A^-_{n+1,i;k}=k(i+1)A^-_{n,i;k}+2k(n-2i)A^-_{n,i-1;k}+(k-1)A^+_{n,i;k},\end{cases}\tag{28}$$

with $A^+_{1,0;k}=1$, $A^+_{1,i;k}=0$ for $i\neq 0$ and $A^-_{1,i;k}=0$ for any i. Clearly, $A^+_{n,i;k}$ and $A^-_{n,i;k}$ are both nonnegative when $k\geqslant 1$. For $n\geqslant 2$, we define

$$A^+_{n;k}(x)=\sum_{i=0}^{\lfloor (n-1)/2\rfloor}A^+_{n,i;k}x^i,\ A^-_{n;k}(x)=\sum_{i=0}^{\lfloor (n-2)/2\rfloor}A^-_{n,i;k}x^i.$$

Multiplying both sides of (28) by x^i and summing over all i, we obtain the recurrence system

$$\begin{cases}A^+_{n+1;k}(x)=(1+2k(n-1)x)A^+_{n;k}(x)+kx(1-4x)\frac{d}{dx}A^+_{n;k}(x)+xA^-_{n;k}(x),\\ A^-_{n+1;k}(x)=k(1+2(n-2)x)A^-_{n;k}(x)+kx(1-4x)\frac{d}{dx}A^-_{n;k}(x)+(k-1)A^+_{n;k}(x),\end{cases}\tag{29}$$

with $A^+_{1;k}(x)=1$ and $A^-_{1;k}(x)=0$. For convenience, we add more details regarding the derivation of (29). Note that $\sum_i A^+_{n+1,i;k}x^i=A^+_{n+1;k}(x)$ and $\sum_i A^-_{n+1,i;k}x^i=A^-_{n+1;k}(x)$. Moreover,

$$\begin{aligned}&\sum_i(1+ki)A^+_{n,i;k}x^i+2k\sum_i(n-2i+1)A^+_{n,i-1;k}x^i+\sum_i A^-_{n,i-1;k}x^i\\&=A^+_{n;k}(x)+kx\frac{d}{dx}A^+_{n;k}(x)+2kx\sum_i(n-2i-1)A^+_{n,i;k}x^i+x\sum_i A^-_{n,i;k}x^i\\&=A^+_{n;k}(x)+kx\frac{d}{dx}A^+_{n;k}(x)+2k(n-1)xA^+_{n;k}(x)-4kx\frac{d}{dx}A^+_{n;k}(x)+xxA^-_{n;k}(x),\end{aligned}$$

$$\begin{aligned}&k\sum_i(i+1)A^-_{n,i;k}x^i+2k\sum_i(n-2i)A^-_{n,i-1;k}x^i+(k-1)\sum_i A^+_{n,i;k}x^i\\&=kx\frac{d}{dx}A^-_{n;k}(x)+kA^-_{n;k}(x)+2kx\sum_i(n-2i-2)A^-_{n,i;k}x^i+(k-1)A^+_{n;k}(x)\\&=kx\frac{d}{dx}A^-_{n;k}(x)+kA^-_{n;k}(x)+2kx(n-2)A^-_{n;k}(x)-4kx^2\frac{d}{dx}A^-_{n;k}(x)+(k-1)A^+_{n;k}(x).\end{aligned}$$

After simplifying the above expressions, we get (29).

Upon substituting $J=Iy$, $u=x+y$ and $v=xy$ in (27), we obtain

$$D^n_{G_0}(I)=Iy\sum_{i=0}^{\lfloor (n-1)/2\rfloor}A^+_{n,i;k}(xy)^i(x+y)^{n-1-2i}+Ixy\sum_{i=0}^{\lfloor (n-2)/2\rfloor}A^-_{n,i;k}(xy)^i(x+y)^{n-2-2i}.\tag{30}$$

It follows from (26) with (30) that

$$A^{(k)}_n(x)=a^{(k)}_n(x)+xb^{(k)}_n(x),\tag{31}$$

where

$$\begin{cases}a^{(k)}_n(x)=\sum_{i\geqslant 0}A^+_{n,i;k}x^i(1+x)^{n-1-2i}=(1+x)^{n-1}A^+_{n;k}\left(\frac{x}{(1+x)^2}\right),\\ b^{(k)}_n(x)=\sum_{i\geqslant 0}A^-_{n,i;k}x^i(1+x)^{n-2-2i}=(1+x)^{n-2}A^-_{n;k}\left(\frac{x}{(1+x)^2}\right).\end{cases}\tag{32}$$

Therefore, the polynomials $A_n^{(k)}(x)$ are bi-γ-positive, and so $A_n(x,q)$ are bi-γ-positive when $q \in [0,1]$. Let $0 \leqslant p \leqslant 1$ and $0 \leqslant q \leqslant 1$ be two given real numbers. Combining (11) and Theorem 2.4, we get that $A_n(x,p,q)$ is alternatingly increasing. This completes the proof. $\square$

Proposition 3.15. *Let $a_n^{(k)}(x)$ and $b_n^{(k)}(x)$ be defined by (32). Then we have*

$$\begin{cases} a_{n+1}^{(k)}(x) & = (1+x+k(n-1)x)a_n^{(k)}(x) + kx(1-x)\frac{\mathrm{d}}{\mathrm{d}x}a_n^{(k)}(x) + xb_n^{(k)}(x), \\ b_{n+1}^{(k)}(x) & = k(1+(n-1)x)b_n^{(k)}(x) + kx(1-x)\frac{\mathrm{d}}{\mathrm{d}x}b_n^{(k)}(x) + (k-1)a_n^{(k)}(x), \end{cases}$$

with $a_1^{(k)}(x) = 1$ and $b_1^{(k)}(x) = 0$.

Proof. According to (32), we see that

$$A_{n;k}^{+}\left(\frac{x}{(1+x)^2}\right) = \frac{a_n^{(k)}(x)}{(1+x)^{n-1}},\ A_{n;k}^{-}\left(\frac{x}{(1+x)^2}\right) = \frac{b_n^{(k)}(x)}{(1+x)^{n-2}}.$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}x}A_{n;k}^{+}\left(\frac{x}{(1+x)^2}\right) = \frac{(1+x)\frac{\mathrm{d}}{\mathrm{d}x}a_n^{(k)}(x) - (n-1)a_n^{(k)}(x)}{(1-x)(1+x)^{n-3}},$$
$$\frac{\mathrm{d}}{\mathrm{d}x}A_{n;k}^{-}\left(\frac{x}{(1+x)^2}\right) = \frac{(1+x)\frac{\mathrm{d}}{\mathrm{d}x}b_n^{(k)}(x) - (n-2)b_n^{(k)}(x)}{(1-x)(1+x)^{n-4}}.$$

Using the replacement $x \to \frac{x}{(1+x)^2}$ in (29), and then substituting the above expressions, we get

$$\frac{a_{n+1}^{(k)}(x)}{(1+x)^n} = \left(1+2k(n-1)\frac{x}{(1+x)^2}\right)\frac{a_n^{(k)}(x)}{(1+x)^{n-1}} + kx(1-x)\frac{(1+x)\frac{\mathrm{d}}{\mathrm{d}x}a_n^{(k)}(x) - (n-1)a_n^{(k)}(x)}{(1+x)^{n+1}} + \frac{x}{(1+x)^2}\frac{b_n^{(k)}(x)}{(1+x)^{n-2}},$$

$$\frac{b_{n+1}^{(k)}(x)}{(1+x)^{n-1}} = k\left(1+2(n-2)\frac{x}{(1+x)^2}\right)\frac{b_n^{(k)}(x)}{(1+x)^{n-2}} + kx(1-x)\frac{(1+x)\frac{\mathrm{d}}{\mathrm{d}x}b_n^{(k)}(x) - (n-2)b_n^{(k)}(x)}{(1+x)^{n}} + (k-1)\frac{a_n^{(k)}(x)}{(1+x)^{n-1}}.$$

Simplifying these two expressions, we get the desired recurrence system. $\square$

By Proposition 3.15, one can see that $b_n^{(k)}(x)$ is divisible by $k-1$ when $n \geqslant 2$. Below the polynomials $a_n^{(k)}(x)$ and $b_n^{(k)}(x)$ for $2 \leqslant n \leqslant 4$:

$$a_2^{(k)}(x) = 1+x,\ b_2^{(k)}(x) = k-1,$$
$$a_3^{(k)}(x) = 1+(3k+1)x+x^2,\ b_3^{(k)}(x) = (k^2-1)(1+x),$$
$$a_4^{(k)}(x) = 1+(1+6k+4k^2)x+(1+6k+4k^2)x^2+x^3,$$
$$b_4^{(k)}(x) = k^3-1+(4k^3+3k^2-6k-1)x+(k^3-1)x^2.$$

4. On the $1/k$-Eulerian polynomials

4.1. Symmetric decompositions

Recall that $A_n^{(k)}(x) = a_n^{(k)}(x) + xb_n^{(k)}(x)$. From (32), we see that

$$a_n^{(k)}(x) = (1+x)^{n-1}A_{n;k}^{+}\left(\frac{x}{(1+x)^2}\right),$$

$$b_n^{(k)}(x) = (1+x)^{n-2} A_{n;k}^{-}\left(\frac{x}{(1+x)^2}\right).$$

Below the polynomials $A_{n;k}^{+}(x)$ and $A_{n;k}^{-}(x)$ for $2 \leqslant n \leqslant 4$:

$$A_{2;k}^{+}(x) = 1,\ A_{2;k}^{-}(x) = k-1,\ A_{3;k}^{+}(x) = 1+(3k-1)x,\ A_{3;k}^{-}(x) = k^2-1,$$
$$A_{4;k}^{+}(x) = 1+(6k+4k^2-2)x,\ A_{4;k}^{-}(x) = k^3-1+(1-6k+3k^2+2k^3)x.$$

In this subsection, we present combinatorial interpretations of $a_n^{(k)}(x)$ and $b_n^{(k)}(x)$.

Let $j^i = \underbrace{j, \ldots, j}_{i}$ for $i, j \geqslant 1$. We say that a permutation of $\{1^k, 2^k, \ldots, n^k\}$ is a *k-Stirling permutation* of order n if for each i, $1 \leqslant i \leqslant n$, all entries between the two occurrences of i are at least i. When $k = 2$, the k-Stirling permutation reduces to the classical Stirling permutation, see [8,30] for instance. Let $\mathcal{Q}_n(k)$ be the set of *k-Stirling permutations* of order n. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{nk} \in \mathcal{Q}_n(k)$. We say that an index i is a *longest ascent plateau* if $\sigma_{i-1} < \sigma_i = \sigma_{i+1} = \sigma_{i+2} = \cdots = \sigma_{i+k-1}$, where $2 \leqslant i \leqslant nk-k+1$. A *longest left ascent plateau* of σ is a longest ascent plateau of σ endowed with a 0 in the front of σ. Let $\mathrm{ap}(\sigma)$ (resp. $\mathrm{lap}(\sigma)$) be the number of longest ascent plateaus (resp. longest left ascent plateaus) of σ. It is clear that

$$\mathrm{lap}(\sigma) := \begin{cases} \mathrm{ap}(\sigma)+1, & \text{if } \sigma_1 = \sigma_2 = \cdots = \sigma_k; \\ \mathrm{ap}(\sigma), & \text{otherwise.} \end{cases}$$

Example 4.1. We have

$$\mathrm{ap}(112\mathbf{3}332\mathbf{4}4421) = 2,\ \mathrm{lap}(111223334442) := \mathrm{lap}(0\mathbf{1}1122\mathbf{3}33\mathbf{4}442) = 3.$$

The following results were obtained in [43]:

$$A_n^{(k)}(x) = \sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{ap}(\sigma)},\ x^n A_n^{(k)}\left(\frac{1}{x}\right) = \sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{lap}(\sigma)}. \tag{33}$$

Note that $\deg A_n^{(k)}(x) = n-1$. Let $\left(a_n^{(k)}(x), b_n^{(k)}(x)\right)$ be the symmetric decomposition of $A_n^{(k)}(x)$. Let $\overline{\mathcal{Q}}_n(k) = \{\sigma \in \mathcal{Q}_n(k) \mid \sigma_j < \sigma_{j+1} \text{ for some j}\in\text{[k-1]}\}$. Put $\mathcal{Q}_n = \mathcal{Q}_n(2)$. Combining (1) and (33), we obtain

$$a_n^{(k)}(x) = \frac{\sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{ap}(\sigma)} - \sum_{\sigma\in\mathcal{Q}_n(k)} x^{\mathrm{lap}(\sigma)}}{1-x}.$$

So we get the following result.

Proposition 4.2. *We have*

$$a_n^{(k)}(x) = \sum_{\substack{\sigma\in\mathcal{Q}_n(k)\\ \sigma_1=\sigma_2=\cdots=\sigma_k}} x^{\mathrm{ap}(\sigma)},\ xb_n^{(k)}(x) = \sum_{\sigma\in\overline{\mathcal{Q}}_n(k)} x^{\mathrm{ap}(\sigma)}.$$

In particular, we have

$$a_n^{(2)}(x) = \sum_{\substack{\sigma\in\mathcal{Q}_n\\ \sigma_1=\sigma_2}} x^{\mathrm{ap}(\sigma)},\ xb_n^{(2)}(x) = \sum_{\substack{\sigma\in\mathcal{Q}_n\\ \sigma_1<\sigma_2}} x^{\mathrm{ap}(\sigma)}.$$

4.2. A combinatorial interpretation of the bi-γ-coefficients of $A_n^{(2)}(x)$

It follows from (23) and (31) that

$$A_n^{(2)}(x) = 2^n A_n(x, 1/2) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{exc}(\pi)} 2^{n-\mathrm{cyc}(\pi)} = a_n^{(2)}(x) + xb_n^{(2)}(x).$$

Below are the symmetric polynomials $a_n^{(2)}(x)$ and $b_n^{(2)}(x)$ for $n \leqslant 4$:

$$a_1^{(2)}(x) = 1,\ b_1^{(2)}(x) = 0,\ a_2^{(2)}(x) = 1 + x,\ b_2^{(2)}(x) = 1,\ a_3^{(2)}(x) = 1 + 7x + x^2,$$
$$b_3^{(2)}(x) = 3 + 3x,\ a_4^{(2)}(x) = 1 + 29x + 29x^2 + x^3,\ b_4^{(2)}(x) = 7 + 31x + 7x^2.$$

We say that $\pi \in \mathcal{S}_n$ is a *circular permutation* if it has only one cycle. Let $A = \{x_1, \ldots, x_j\}$ be a finite set of positive integers, and let $\mathcal{C}_A$ be the set of all circular permutations of A. Let $w \in \mathcal{C}_A$. We will always write w by using its canonical presentation $w = y_1y_2\cdots y_j$, where $y_1 = \min A$, $y_i = w^{i-1}(y_1)$ for $2 \leqslant i \leqslant j$ and $y_1 = w^j(y_1)$. A *cycle peak* (resp. *cycle double ascent*, *cycle double descent*) of w is an entry y_i, $2 \leqslant i \leqslant j$, such that $y_{i-1} < y_i > y_{i+1}$ (resp. $y_{i-1} < y_i < y_{i+1}$, $y_{i-1} > y_i > y_{i+1}$), where we set $y_{j+1} = \infty$, i.e., y_{j+1} is the positive infinity. As shown in [41, p. 10–11], the motivation of setting $y_{j+1} = \infty$ lies in the fact that when set $y_{j+1} = \infty$, we can gave a combinatorial interpretation of the q-alternating run polynomials, which are defined by the following recurrence relation:

$$R_{n+1}(x,q) = (q + nx)xR_n(x,q) + x(1-x^2)\frac{\partial}{\partial x}R_n(x,q),\ R_0(x,q) = 1.$$

In this subsection, we use the same assumption, i.e., put a ∞ at the end of each cycle of π.

Let $\operatorname{cpk}(w)$ be the number of cycle peaks of the circular permutation w. A *run* of w is a maximal consecutive subsequence that is increasing or decreasing. Following [41], the number of *cycle runs* $\operatorname{crun}(w)$ of w is defined to be the number of runs of the word $y_1y_2\cdots y_j\infty$. Assume that $\operatorname{cyc}(\pi) = s$ and $\pi = w_1w_2\cdots w_s$, where w_i is the ith cycle of π. The numbers of *cycle runs* and *cycle peaks* of π are respectively defined by

$$\operatorname{crun}(\pi) = \sum_{i=1}^{s}\operatorname{crun}(w_i),\ \operatorname{cpk}(\pi) = \sum_{i=1}^{s}\operatorname{cpk}(w_i).$$

For $\pi \in \mathcal{S}_n$, it is clear that $1 \leqslant \operatorname{crun}(\pi) \leqslant n$.

Example 4.3. We have

$$\operatorname{crun}\ ((1)(2)(3)\cdots(n)) := \operatorname{crun}\ ((1\infty)(2\infty)(3\infty)\cdots(n\infty)) = n,$$

$$\operatorname{crun}\ ((1,2,3,\ldots,n)) := \operatorname{crun}\ ((1,2,3,\ldots,n,\infty)) = 1.$$

Example 4.4. If $\pi = (1,4,2)(3,5,6)(7) \in \mathcal{S}_7$, then $\operatorname{crun}(\pi) = 5$, since the numbers of cycle runs in the three cycles are 3, 1, 1, respectively. Moreover, $\operatorname{cpk}\ ((1,\mathbf{4},2)(3,5,6)(7)) = 1$, so we get $\operatorname{crun}(\pi) = 2\operatorname{cpk}(\pi) + \operatorname{cyc}(\pi) = 2 + 3 = 5$.

Proposition 4.5. *For any* $\pi \in \mathcal{S}_n$*, we have* $\operatorname{crun}(\pi) = 2\operatorname{cpk}(\pi) + \operatorname{cyc}(\pi)$.

Proof. Assume that $\pi = w_1w_2\cdots w_s$, where w_i is the ith cycle of π. Since we put a ∞ at the end of w_i, then w_i starts in an ascending run and ends in an ascending run. So we get $\operatorname{crun}(w_i) = 2\operatorname{cpk}(w_i) + 1$, which yields that

$$\operatorname{crun}(\pi) = \sum_{i=1}^{s}\operatorname{crun}(w_i) = 2\sum_{i=1}^{s}\operatorname{cpk}(w_i) + s = 2\operatorname{cpk}(\pi) + \operatorname{cyc}(\pi). \quad \square$$

We can now conclude the following result.

Theorem 4.6. *Let* $\mathcal{S}_{n,i}$ *be the set of permutations in* $\mathcal{S}_n$ *with i cycle runs. For* $n \geqslant 2$*, we have*

$$A_n^{(2)}(x) = \sum_{\pi \in \mathcal{S}_n} x^{\operatorname{exc}(\pi)}2^{n-\operatorname{cyc}(\pi)} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \xi_{n,i}^{+}x^i(1+x)^{n-1-2i} + x\sum_{j=0}^{\lfloor (n-2)/2 \rfloor} \xi_{n,j}^{-}x^j(1+x)^{n-2-2j},$$

where

$$\begin{cases} \xi_{n,i}^{+} = \sum_{\pi\in\mathcal{S}_{n,2i+1}} 2^{\mathrm{crun}\,(\pi)-\mathrm{cyc}\,(\pi)} = \sum_{\pi\in\mathcal{S}_{n,2i+1}} 4^{\mathrm{cpk}\,(\pi)}, \\ \xi_{n,j}^{-} = \sum_{\pi\in\mathcal{S}_{n,2j+2}} 2^{\mathrm{crun}\,(\pi)-\mathrm{cyc}\,(\pi)} = \sum_{\pi\in\mathcal{S}_{n,2j+2}} 4^{\mathrm{cpk}\,(\pi)}. \end{cases} \quad (34)$$

Proof. Note that $2^2A_2(x, 1/2) = 1 + x + x$ and $2^3A_3(x, 1/2) = [(1 + x)^2 + 5x] + 3x(1 + x)$. Thus $\xi_{2,0}^{+} = \xi_{2,0}^{-} = 1,\ \xi_{3,0}^{+} = 1,\ \xi_{3,1}^{+} = 5,\ \xi_{3,0}^{-} = 3$. Note that $\mathcal{S}_{2,1} = \{(12)\},\ \mathcal{S}_{2,2} = \{(1)(2)\}$,

$$\mathcal{S}_{3,1} = \{(1, 2, 3)\},\ \mathcal{S}_{3,2} = \{(1, 2)(3), (1, 3)(2), (1)(2, 3)\}, \mathcal{S}_{3,3} = \{(1)(2)(3), (1, 3, 2)\}.$$

It is easy to check that (34) holds for $n = 2, 3$. We proceed by induction on n. In order to get permutations in $\mathcal{S}_{n+1,2i+1}$, we distinguish among three distinct cases:

- (c_1) if $\pi \in \mathcal{S}_{n,2i}$, then we can insert $n + 1$ into π as a new cycle. This gives the term $\xi_{n,i-1}^{-}$;
- (c_2) if $\pi \in \mathcal{S}_{n,2i+1}$, then $2\mathrm{cpk}\,(\pi) + \mathrm{cyc}\,(\pi) = 2i + 1$. We can insert $n + 1$ just before or right after each cycle peak of π. Moreover, we can insert $n + 1$ at the end of a cycle of π. This gives the term $(2i + 1)\xi_{n,i}^{+}$;
- (c_3) if $\pi \in \mathcal{S}_{n,2i-1}$, then $2\mathrm{cpk}\,(\pi) + \mathrm{cyc}\,(\pi) = 2i - 1$. We can insert $n + 1$ into any of the remaining $n - (2i - 1)$ positions, and the number of cycle runs is increased by two. This gives the term $4(n-2i+1)\xi_{n,i-1}^{+}$. As an illustration, consider $\pi = (1, 3, 5, 2)(4, 6)(7) \in \mathcal{S}_{7,5}$. When 8 is inserted right after 1 or 4, the number of cycle runs will increased by two.

Similarly, there are three ways to get permutations in $\mathcal{S}_{n+1,2i+2}$ by inserting the entry $n + 1$:

- (c_1) if $\pi \in \mathcal{S}_{n,2i+1}$, then we can insert $n + 1$ into π as a new cycle. This gives the term $\xi_{n,i}^{+}$;
- (c_2) if $\pi \in \mathcal{S}_{n,2i+2}$, then $2\mathrm{cpk}\,(\pi) + \mathrm{cyc}\,(\pi) = 2i + 2$. We can insert $n + 1$ just before or right after each cycle peak of π. Moreover, we can insert $n + 1$ at the end of a cycle of π. This gives the term $(2i + 2)\xi_{n,i}^{-}$;
- (c_3) if $\pi \in \mathcal{S}_{n,2i}$, then $2\mathrm{cpk}\,(\pi) + \mathrm{cyc}\,(\pi) = 2i$. We can insert $n + 1$ into any of the remaining $n - 2i$ positions, and the number of cycle runs is increased by two. This gives the term $4(n-2i)\xi_{n,i-1}^{-}$.

In conclusion, we have

$$\begin{cases} \xi_{n+1,i}^{+} = (2i + 1)\xi_{n,i}^{+} + 4(n - 2i + 1)\xi_{n,i-1}^{+} + \xi_{n,i-1}^{-}, \\ \xi_{n+1,i}^{-} = (2i + 2)\xi_{n,i}^{-} + 4(n - 2i)\xi_{n,i-1}^{-} + \xi_{n,i}^{+}. \end{cases}$$

Comparing this with (28) leads to $\xi_{n,i}^{+} = A_{n,i;2}^{+}$ and $\xi_{n,i}^{-} = A_{n,i;2}^{-}$, and this completes the proof. □

5. Excedance-type polynomials of signed permutations

5.1. Basic definitions

Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) \in \mathcal{S}_n^B$. It should be noted that the n letters appearing in the cycle notation of $\sigma \in \mathcal{S}_n^B$ are the letters $\sigma(1), \sigma(2), \ldots, \sigma(n)$. We say that i is an *excedance* (resp. *anti-excedance*, *fixed point*, *singleton*) of σ if $\sigma(|\sigma(i)|) > \sigma(i)$ (resp. $\sigma(|\sigma(i)|) < \sigma(i)$, $\sigma(i) = i$, $\sigma(i) = \bar{i}$). Let $\mathrm{exc}\,(\sigma)$ (resp. $\mathrm{aexc}\,(\sigma)$, $\mathrm{fix}\,(\sigma)$, $\mathrm{single}\,(\sigma)$, $\mathrm{neg}\,(\sigma)$) be the number of excedances (resp. anti-excedances, fixed points, singletons, negative entries) of σ.

Example 5.1. The signed permutation $\pi = \bar{3}51\bar{7}24689\bar{9}$ can be written as $(\bar{9})(\bar{3}, 1)(2, 5)(4, \bar{7}, 6)(8)$. Thus, π with only one singleton 9 and one fixed point 8, and π has 3 excedances, 4 anti-excedances and 3 negative entries.

We say that $i \in [n]$ is a *weak excedance* of σ if $\sigma(i) = i$ or $\sigma(|\sigma(i)|) > \sigma(i)$ (see [13, p. 431]). Let $\mathrm{wexc}\,(\sigma)$ be the number of weak excedances of σ. Then $\mathrm{wexc}\,(\sigma) = \mathrm{exc}\,(\sigma) + \mathrm{fix}\,(\sigma)$. The number of *type B descent* of σ is defined by

$$\mathrm{des}_B(\sigma) = \#\{i \in \{0, 1, \ldots, n - 1\} \mid \sigma(i) > \sigma(i + 1)\},$$

where $\sigma(0) := 0$. Following [13, Theorem 3.15], the statistics des_B and wexc have the same distribution over B_n, and their common enumerative polynomial is the *type B Eulerian polynomial* (see [52, A060187]):

$$B_n(x) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\mathrm{des}_B(\sigma)} = \sum_{\sigma \in \mathcal{S}_n^B} x^{\mathrm{wexc}\,(\sigma)}. \tag{35}$$

Let $Q(n, i)$ be the number of permutations in $\mathcal{S}_n$ with i left peaks (see [52, A008971]). Using the theory of enriched P-partitions, Petersen [46, Proposition 4.15] obtained that

$$B_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} 4^i Q(n, i) x^i (1+x)^{n-2i}, \tag{36}$$

which has been extensively studied, see [21,37,51,55] and the references therein.

5.2. A unified generalization of the expansion (36) and Propositions 3.1 and 3.2

Consider the following polynomials

$$B_n(x, y, s, t, p, q) = \sum_{\sigma \in \mathcal{S}_n^B} x^{\mathrm{exc}\,(\sigma)} y^{\mathrm{aexc}\,(\sigma)} s^{\mathrm{single}\,(\sigma)} t^{\mathrm{fix}\,(\sigma)} p^{\mathrm{neg}\,(\sigma)} q^{\mathrm{cyc}\,(\sigma)}.$$

Theorem 5.2. *We have*

$$B_n(x, y, s, t, p, q) = (1+p)^n y^n A_n\left(\frac{x}{y}, \frac{t+sp}{y+py}, q\right). \tag{37}$$

In the sequel, we shall prove Theorem 5.2.

Lemma 5.3. *Let p and q be two given parameters. If*

$$G_3 = \{J \to qJ(t+sp), s \to (1+p)xy, t \to (1+p)xy, x \to (1+p)xy, y \to (1+p)xy\}, \tag{38}$$

then we have $D_{G_3}^n(J) = JB_n(x, y, s, t, p, q)$.

Proof. We first introduce a grammatical labeling of $\sigma \in \mathcal{S}_n^B$ as follows:

(L_1) if i is an excedance, then put a superscript label x right after $\sigma(i)$;
(L_2) if i is an anti-excedance, then put a superscript label y right after $\sigma(i)$;
(L_3) if i is a fixed point, then put a superscript label t right after i;
(L_4) if i is a singleton, then put a superscript label s right after $\bar{i}$;
(L_5) put a superscript label J at the end of σ;
(L_6) put a subscript label q at the end of each cycle of σ;
(L_7) put a subscript label p right after each negative entry of σ.

For example, for $\sigma = (1, 3, \overline{2}, 6)(\overline{4})(5)$, the grammatical labeling of σ is given below:

$$(1^x 3^y \overline{2}_p^x 6^y)_q (\overline{4}_p^s)_q (5^t)_q^J.$$

Note that the weight of σ is given by

$$w(\sigma) = Jx^{\mathrm{exc}\,(\sigma)} y^{\mathrm{aexc}\,(\sigma)} s^{\mathrm{single}\,(\sigma)} t^{\mathrm{fix}\,(\sigma)} p^{\mathrm{neg}\,(\sigma)} q^{\mathrm{cyc}\,(\sigma)}.$$

For $n = 1$, we have $\mathcal{S}_1^B = \{(1^t)_q^J, (\overline{1}_p^s)_q^J\}$. Note that $D_{G_3}(J) = qJ(t+sp)$. Hence the result holds for $n = 1$. We proceed by induction. Suppose we get all labeled permutations in $\sigma \in \mathcal{S}_{n-1}^B$, where $n \geqslant 2$. Let $\widehat{\sigma}$ be obtained from $\sigma \in \mathcal{S}_{n-1}^B$ by inserting the entry n or $\overline{n}$. There are five ways to label the inserted element and relabel some elements of σ:

(c_1) if n or $\overline{n}$ appended as a new cycle, then the changes of labeling are illustrated as follows:

$$\cdots(\cdots)_q^J \to \cdots(\cdots)_q(n^t)_q^J, \quad \cdots(\cdots)^J \to \cdots(\cdots)_q(\overline{n}_p^s)_q^J;$$

(c_2) if we insert n or $\overline{n}$ right after a fixed point, then the changes of labeling are illustrated as follows:

$$\cdots(i^t)_q(\cdots)\cdots \to \cdots(i^x n^y)_q(\cdots)\cdots, \quad \cdots(i^t)_q(\cdots)\cdots \to \cdots(i^y\overline{n}_p^x)_q(\cdots)\cdots;$$

(c_3) if we insert n or $\overline{n}$ right after a singleton, then the changes of labeling are illustrated as follows:

$$\cdots(\bar{i}_p^s)_q(\cdots)\cdots \to \cdots(\bar{i}_p^x n^y)_q(\cdots)\cdots, \quad \cdots(\bar{i}_p^s)_q(\cdots)\cdots \to \cdots(\bar{i}_p^y\overline{n}_p^x)_q(\cdots)\cdots;$$

(c_4) if we insert n or $\overline{n}$ right after an excedance, then the changes of labeling are illustrated as follows:

$$\cdots(\cdots\sigma(i)^x\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots \to \cdots(\cdots\sigma(i)^x n^y\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots,$$

$$\cdots(\cdots\sigma(i)^x\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots \to \cdots(\cdots\sigma(i)^y\overline{n}_p^x\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots;$$

(c_5) if we insert n or $\overline{n}$ right after an anti-excedance, then the changes of labeling are illustrated as follows:

$$\cdots(\cdots\sigma(i)^y\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots \to \cdots(\cdots\sigma(i)^x n^y\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots,$$

$$\cdots(\cdots\sigma(i)^y\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots \to \cdots(\cdots\sigma(i)^y\overline{n}_p^x\sigma(|\sigma(i)|)\cdots)_q(\cdots)\cdots.$$

In each case, the insertion of n or $\overline{n}$ corresponds to one substitution rule in G. Therefore, the action of D_{G_3} on the set of weighted signed permutations in $\mathcal{S}_{n-1}^B$ gives the set of weighted signed permutations in $\mathcal{S}_n^B$. This yields the desired result. □

A proof of Theorem 5.2. Let G_3 be the grammar given in Lemma 5.3. Consider a change of the grammar G_3. Setting $A = t + sp$, $B = (1 + p)x$ and $C = (1 + p)y$, we get

$$D_{G_3}(J) = qJA,\ D_{G_3}(A) = BC,\ D_{G_3}(B) = BC,\ D_{G_3}(C) = BC.$$

Let $G_4 = \{J \to qJA, A \to BC, B \to BC, C \to BC\}$. It follows from Lemma 3.12 that

$$D_{G_4}^n(J) = J\sum_{\pi\in\mathcal{S}_n} A^{\mathrm{fix}(\pi)}B^{\mathrm{exc}(\pi)}C^{\mathrm{drop}(\pi)}q^{\mathrm{cyc}(\pi)}.$$

Upon substituting $A = t + sp$, $B = (1 + p)x$ and $C = (1 + p)y$ in the above expansion, we get

$$D_{G_3}^n(J) = J\sum_{\pi\in\mathcal{S}_n}(t+sp)^{\mathrm{fix}(\pi)}(x+px)^{\mathrm{exc}(\pi)}(y+py)^{\mathrm{drop}(\pi)}q^{\mathrm{cyc}(\pi)}. \tag{39}$$

It follows from (39) and Proposition 3.11 that

$$D_{G_3}^n(J) = J(1+p)^n y^n A_n\left(\frac{x}{y}, \frac{t+sp}{y+py}, q\right).$$

Combining this with Lemma 5.3, we obtain (37). This completes the proof. □

Combining (11) and (37), we get the following result.

Corollary 5.4. *Let $\mathcal{S}_{n,i,j} = \{\pi \in \mathcal{S}_n : \mathrm{cda}(\pi) = 0,\ \mathrm{fix}(\pi) = i,\ \mathrm{exc}(\pi) = j\}$, and set*

$$\gamma_{n,i,j}(q) = \sum_{\pi\in\mathcal{S}_{n,i,j}} q^{\mathrm{cyc}(\pi)}.$$

Then we have

$$B_n(x,y,s,t,p,q)=\sum_{i=0}^{n}(t+sp)^i(1+p)^{n-i}\sum_{j=0}^{\lfloor (n-i)/2\rfloor}\gamma_{n,i,j}(q)(xy)^j(x+y)^{n-i-2j}.$$

In particular, setting $y=s=1$ *and* $t=0$ *in* $B_n(x,y,s,t,p,q)$, *we have*

$$\sum_{\sigma\in\mathcal{D}_n^B}x^{\mathrm{exc}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}q^{\mathrm{cyc}\,(\pi)}=\sum_{i=0}^{n}p^i(1+p)^{n-i}\sum_{j=0}^{\lfloor (n-i)/2\rfloor}\gamma_{n,i,j}(q)x^j(x+1)^{n-i-2j}. \tag{40}$$

When $p=0$, then (40) reduces to (7). From (37), we get

$$B_n(x,1,s,t,1,q)=\sum_{\sigma\in\mathcal{S}_n^B}x^{\mathrm{exc}\,(\pi)}s^{\mathrm{single}\,(\sigma)}t^{\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}=2^nA_n\left(x,\frac{t+s}{2},q\right). \tag{41}$$

Note that $B_n(x,1,1,-1,1,1)=2^nd_n(x)$, $B_n(x,1,1,0,1,1)=d_n^B(x)$, $B_n(x,1,2,0,1,1)=2^nA_n(x)$. Combining Theorem 3.6 and (41), we get the following result.

Corollary 5.5. *Let* t,s *and* q *be three given real numbers satisfying* $0\leqslant t+s\leqslant 2$ *and* $0\leqslant q\leqslant 1$. *Then* $B_n(x,1,s,t,1,q)$ *are alternatingly increasing for* $n\geqslant 1$. *In particular, the polynomials* $B_n(x,1,s,2-s,1,q)$ *are bi-*γ*-positive.*

5.3. Several convolution formulas

Consider the following multivariate polynomials

$$B_n(x,y,s,t,p,1)=\sum_{\sigma\in\mathcal{S}_n^B}x^{\mathrm{exc}\,(\sigma)}y^{\mathrm{aexc}\,(\sigma)}s^{\mathrm{single}\,(\sigma)}t^{\mathrm{fix}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}.$$

Set $B_0(x,y,s,t,p,1)=1$. Using (37), we obtain

$$\begin{aligned}B(x,y,s,t,p,1;z)&=\sum_{n=0}^{\infty}B_n(x,y,s,t,p,1)\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}(1+p)^ny^nA_n\left(\frac{x}{y},\frac{t+sp}{y+py},1\right)\frac{z^n}{n!}\\&=\sum_{n=0}^{\infty}A_n\left(\frac{x}{y},\frac{t+sp}{y+py},1\right)\frac{((1+p)yz)^n}{n!}.\end{aligned}$$

Combining this with (10), we get

$$B(x,y,s,t,p,1;z)=\frac{(y-x)\mathrm{e}^{(t+sp)z}}{y\mathrm{e}^{(1+p)xz}-x\mathrm{e}^{(1+p)yz}}. \tag{42}$$

Define

$$\Phi_n(x,y)=xy\frac{x^{n-1}-y^{n-1}}{x-y}=xy(x^{n-2}+x^{n-3}y+\cdots+xy^{n-3}+y^{n-2})\ \text{ for } n\geqslant 2.$$

In particular, $\Phi_2(x,y)=xy$ and $\Phi_3(x,y)=xy(x+y)$. Set $\Phi_0(x,y)=\Phi_1(x,y)=0$.

Theorem 5.6. *For* $n\geqslant 2$, *we have*

$$B_n(x,y,s,t,p,1)=(t+sp)^n+\sum_{k=0}^{n-2}\binom{n}{k}B_k(x,y,s,t,p,1)\Phi_{n-k}(x,y)(1+p)^{n-k}. \tag{43}$$

Proof. Note that

$$\begin{aligned}\Phi(x,y,p;z):&=\sum_{n=0}^{\infty}\Phi_n(x,y)(1+p)^n\frac{z^n}{n!}\\&=\frac{y}{x-y}\sum_{n=2}^{\infty}\frac{((1+p)xz)^n}{n!}-\frac{x}{x-y}\sum_{n=2}^{\infty}\frac{((1+p)yz)^n}{n!}\\&=\frac{y}{x-y}\left(e^{(1+p)xz}-1-(1+p)xz\right)-\frac{x}{x-y}\left(e^{(1+p)yz}-1-(1+p)yz\right)\\&=\frac{y}{x-y}\left(e^{(1+p)xz}-1\right)-\frac{x}{x-y}\left(e^{(1+p)yz}-1\right)\\&=1-\frac{y}{y-x}e^{(1+p)xz}+\frac{x}{y-x}e^{(1+p)yz}.\end{aligned}$$

So we have

$$\Phi(x,y,p;z):=\sum_{n=0}^{\infty}\Phi_n(x,y)(1+p)^n\frac{z^n}{n!}=1-\frac{ye^{(1+p)xz}-xe^{(1+p)yz}}{y-x}.$$

For $n\geqslant 2$, we define

$$f_n(x,y,s,t,p)=(t+sp)^n+\sum_{k=0}^{n-2}\binom{n}{k}B_k(x,y,s,t,p,1)\Phi_{n-k}(x,y)(1+p)^{n-k}. \tag{44}$$

Set $f_0(x,y,s,t,p)=1$ and $f_1(x,y,s,t,p)=t+sp$. It follows from (44) that

$$\begin{aligned}f(x,y,s,t,p;z)&=\sum_{n=0}^{\infty}f_n(x,y,s,t,p)\frac{z^n}{n!}\\&=e^{(t+sp)z}+B(x,y,s,t,p,1;z)\Phi(x,y,p;z)\\&=e^{(t+sp)z}+\left(\frac{(y-x)e^{(t+sp)z}}{ye^{(1+p)xz}-xe^{(1+p)yz}}\right)\left(1-\frac{ye^{(1+p)xz}-xe^{(1+p)yz}}{y-x}\right)\\&=B(x,y,s,t,p,1;z).\end{aligned}$$

Thus we obtain

$$B(x,y,s,t,p,1;z)=e^{(t+sp)z}+B(x,y,s,t,p,1;z)\Phi(x,y,p;z).$$

Equating the coefficients of $\frac{z^n}{n!}$ in both sides of the above expression, we get the desired result. □

Note that

$$A_n(x)=B_n(x,1,0,1,0,1),\ \ d_n(x)=B_n(x,1,0,0,0,1),$$

$$B_n(x)=B_n(x,1,1,x,1,1),\ \ d_n^B(x)=B_n(x,1,1,0,1,1).$$

The following corollary is immediate from Theorem 5.6 by special parametrizations.

Corollary 5.7. *For $n\geqslant 2$, we have*

$$\begin{aligned}A_n(x)&=1+\sum_{k=0}^{n-2}\binom{n}{k}A_k(x)(x+x^2+\cdots+x^{n-1-k}),\\d_n(x)&=\sum_{k=0}^{n-2}\binom{n}{k}d_k(x)(x+x^2+\cdots+x^{n-1-k}),\end{aligned}$$

$$B_n(x) = (1+x)^n + \sum_{k=0}^{n-2} \binom{n}{k} B_k(x)(x + x^2 + \cdots + x^{n-1-k})2^{n-k},$$
$$d_n^B(x) = 1 + \sum_{k=0}^{n-2} \binom{n}{k} d_k^B(x)(x + x^2 + \cdots + x^{n-1-k})2^{n-k}.$$

Using the theory of geometric combinatorics, Juhnke-Kubitzke etal. [35, Corollary 4.2] obtained the convolution formula of $d_n(x)$ that is given in Corollary 5.7. It would be interesting to derive the other convolution formulas by using the theory of geometric combinatorics.

5.4. A relationship between derangement polynomials of types A and B

Consider the following polynomials

$$d_n^B(x,p) = B_n(x,1,1,0,p,1) = \sum_{\sigma\in\mathcal{D}_n^B} x^{\mathrm{exc}\,(\sigma)} p^{\mathrm{neg}\,(\sigma)}.$$

By Theorem 5.2, we get

$$\sum_{n=0}^{\infty} d_n^B(x,p)\frac{z^n}{n!} = \sum_{n=0}^{\infty} A_n\left(x, \frac{p}{1+p}, 1\right)\frac{(1+p)^n z^n}{n!}. \tag{45}$$

From (10), we see that

$$\sum_{n=0}^{\infty} A_n(x,p,1)\frac{z^n}{n!} = \frac{(1-x)\mathrm{e}^{pz}}{\mathrm{e}^{xz} - x\mathrm{e}^{z}}. \tag{46}$$

Combining (45) and (46), we get

$$\sum_{n=0}^{\infty} d_n^B(x,p)\frac{z^n}{n!} = \frac{(1-x)e^{pz}}{e^{(1+p)xz} - xe^{(1+p)z}}. \tag{47}$$

Note that $d_n^B(x,0) = d_n(x)$ and $d_n^B(x,1) = d_n^B(x)$. Below are $d_n^B(x,p)$ for $0 \leqslant n \leqslant 4$:

$$d_0^B(x,p) = 1,\ d_1^B(x,p) = p,\ d_2^B(x,p) = x + 2px + p^2(1+x),$$
$$d_3^B(x,p) = x(1+x) + 3px(2+x) + 3p^2x(3+x) + p^3(1+4x+x^2),$$
$$d_4^B(x,p) = x(1+7x+x^2) + 4px(2+8x+x^2) + 6p^2x(4+9x+x^2) +$$
$$4p^3x(7+10x+x^2) + p^4(1+11x+11x^2+x^3).$$

Lemma 5.8. *For $1 \leqslant i \leqslant n$, let $\widetilde{\mathcal{D}}_{n,i}^B$ be the set of type B derangements of order n with the restriction that the set of negative entries of each derangement is $\{\overline{n}, \overline{n-1}, \ldots, \overline{n-i+1}\}$. Let $d_{n,i}^B(x)$ be the derangement polynomials over $\widetilde{\mathcal{D}}_{n,i}^B$. In other words,*

$$d_{n,i}^B(x) = \sum_{\sigma\in\widetilde{\mathcal{D}}_{n,i}^B} x^{\mathrm{exc}\,(\sigma)}.$$

Then $d_{n,i}^B(x) = d_{n,i-1}^B(x) + d_{n-1,i-1}^B(x)$ for any $1 \leqslant i \leqslant n$, where $d_{n,0}^B(x) = d_n(x)$.

Proof. For any $1 \leqslant i \leqslant n$, we partition the set $\widetilde{\mathcal{D}}_{n,i}^B$ into three subsets:

$$\widetilde{\mathcal{D}}_{n,i}^{B,1} = \{\sigma \in \widetilde{\mathcal{D}}_{n,i}^B \mid \overline{n} \text{ is a singleton of } \sigma\},$$
$$\widetilde{\mathcal{D}}_{n,i}^{B,2} = \{\sigma \in \widetilde{\mathcal{D}}_{n,i}^B \mid \mathrm{single}\,(\sigma) = 0\},$$
$$\widetilde{\mathcal{D}}_{n,i}^{B,3} = \{\sigma \in \widetilde{\mathcal{D}}_{n,i}^B \mid \mathrm{single}\,(\sigma) > 0 \text{ and } \overline{n} \text{ is not a singleton of } \sigma\}.$$

Claim 1. *For $\sigma \in \widetilde{\mathcal{D}}^{B,1}_{n,i}$, we define a bijection $\phi_1 : \widetilde{\mathcal{D}}^{B,1}_{n,i} \mapsto \widetilde{\mathcal{D}}^{B}_{n-1,i-1}$ by deleting the cycle $(\overline{n})$ in σ. Clearly, $\phi_1(\sigma) \in \widetilde{\mathcal{D}}^{B}_{n-1,i-1}$. On the other hand, for $\sigma' \in \widetilde{\mathcal{D}}^{B}_{n-1,i-1}$, the permutation $\phi_1^{-1}(\sigma')$ is obtained from σ' by appending $(\overline{n})$ to σ' as a new cycle.*

Claim 2. *There is an order-preserving bijection $\phi_2 : \widetilde{\mathcal{D}}^{B,2}_{n,i} \mapsto \widetilde{\mathcal{D}}^{B,2}_{n,i-1}$. For $\sigma \in \widetilde{\mathcal{D}}^{B,2}_{n,i}$, we define the map ϕ_2 by*

$$\phi_2(\sigma)(j) = \begin{cases} \sigma(j)+1, & \text{if } \sigma(j) \in \{1,2,\ldots,n-i\}; \\ 1, & \text{if } \sigma(j) = \overline{n-i+1}; \\ \sigma(j), & \text{if } \sigma(j) \in \{\overline{n-i+2},\ldots,\overline{n-1},\overline{n}\}. \end{cases}$$

It is clear that $\phi_2(\sigma) \in \widetilde{\mathcal{D}}^{B,2}_{n,i-1}$ and $\operatorname{exc}(\sigma) = \operatorname{exc}(\phi_2(\sigma))$. For $\sigma' \in \widetilde{\mathcal{D}}^{B,2}_{n,i-1}$, the inverse of ϕ_2 is given as follows:

$$\phi_2^{-1}(\sigma')(j) = \begin{cases} \sigma(j)-1, & \text{if } \sigma(j) \in \{2,3,\ldots,n-i+1\}; \\ \overline{n-i+1}, & \text{if } \sigma'(j) = 1; \\ \sigma(j), & \text{if } \sigma(j) \in \{\overline{n-i+2},\ldots,\overline{n-1},\overline{n}\}. \end{cases}$$

See Example 5.9 for an illustration of ϕ_2.

Claim 3. *There is an order-preserving bijection $\phi_3 : \widetilde{\mathcal{D}}^{B,3}_{n,i} \mapsto \widetilde{\mathcal{D}}^{B}_{n,i-1} \setminus \widetilde{\mathcal{D}}^{B,2}_{n,i-1}$. For $\sigma \in \widetilde{\mathcal{D}}^{B,3}_{n,i}$, let $\operatorname{Single}(\sigma)$ be the set of singletons of σ. Let the set of singletons of $\phi_3(\sigma)$ be defined by*

$$\operatorname{Single}\ (\phi_3(\sigma)) = \{\overline{k+1} : \overline{k} \in \operatorname{Single}(\sigma)\}.$$

Define

$$\mathcal{A}(\sigma) = \{\overline{n}, \overline{n-1}, \ldots, \overline{n-i+1}\} \cup \{1,2,\ldots,n-i\} \setminus \operatorname{Single}(\sigma),$$
$$\mathcal{B}(\sigma) = \{\overline{n}, \overline{n-1}, \ldots, \overline{n-i+2}\} \cup \{1,2,\ldots,n-i,n-i+1\} \setminus \operatorname{Single}\ (\phi_3(\sigma)).$$

We write the elements in $\mathcal{A}(\sigma)$ and $\mathcal{B}(\sigma)$ in increasing order. If $\sigma(j)$ is the kth element of $\mathcal{A}(\sigma)$, then let $\phi_3(\sigma)(j)$ be the kth element of $\mathcal{B}(\sigma)$. It is clear that $\phi_3(\sigma)$ has at least one singleton. See Example 5.10 for instance. Along the same lines, one can define the inverse of ϕ_3. It should be noted that the order-preserving bijection ϕ_3 does not change the number of excedances.

In conclusion, we have

$$\begin{aligned} \sum_{\sigma \in \widetilde{\mathcal{D}}^{B}_{n,i}} x^{\operatorname{exc}(\sigma)} &= \sum_{\sigma \in \widetilde{\mathcal{D}}^{B,1}_{n,i}} x^{\operatorname{exc}(\sigma)} + \sum_{\sigma \in \widetilde{\mathcal{D}}^{B,2}_{n,i}} x^{\operatorname{exc}(\sigma)} + \sum_{\sigma \in \widetilde{\mathcal{D}}^{B,3}_{n,i}} x^{\operatorname{exc}(\sigma)} \\ &= \sum_{\sigma \in \widetilde{\mathcal{D}}^{B}_{n-1,i-1}} x^{\operatorname{exc}(\sigma)} + \sum_{\sigma \in \widetilde{\mathcal{D}}^{B}_{n,i-1}} x^{\operatorname{exc}(\sigma)}, \end{aligned}$$

which leads to the desired result. □

Example 5.9. Let $\sigma = (1,4,3,\overline{9},\overline{8})(2,5)(\overline{6},\overline{7}) \in \widetilde{\mathcal{D}}^{B,2}_{9,4}$. Then $\phi_2(\sigma) = (2,5,4,\overline{9},\overline{8})(3,6)(1,\overline{7})$.

Example 5.10. Let $\sigma = (1,4,3,\overline{9},\overline{8})(2,5)(\overline{6})(\overline{7}) \in \widetilde{\mathcal{D}}^{B,3}_{9,4}$. Then

$$\operatorname{Single}(\sigma) = \{\overline{6},\overline{7}\},\ \operatorname{Single}(\phi_3(\sigma)) = \{\overline{7},\overline{8}\}.$$

Moreover, $\mathcal{A}(\sigma) = \{\overline{9},\overline{8},1,2,3,4,5\}$, $\mathcal{B}(\sigma) = \{\overline{9},1,2,3,4,5,6\}$. Then the order-preserving bijection between $\mathcal{A}(\sigma)$ and $\mathcal{B}(\sigma)$ can be illustrated by the following array:

$$\begin{pmatrix} \overline{9} & \overline{8} & 1 & 2 & 3 & 4 & 5 \\ \overline{9} & 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Therefore, $\phi_3(\sigma) = (2,5,4,\overline{9},1)(3,6)(\overline{7})(\overline{8})$.

We can now conclude the following result.

Theorem 5.11. *We have*

$$d_n^B(x,p)=\sum_{\sigma\in\mathcal{D}_n^B}x^{\operatorname{exc}(\sigma)}p^{\operatorname{neg}(\sigma)}=\sum_{i=0}^{n}\binom{n}{i}p^i\sum_{j=0}^{i}\binom{i}{j}d_{n-j}(x).$$

In particular,

$$d_n^B(x)=\sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}d_{n-j}(x).$$

Proof. Iterating the formula $d_{n,i}^B(x)=d_{n,i-1}^B(x)+d_{n-1,i-1}^B(x)$ leads to

$$d_{n,i}^B=\sum_{j=0}^{i}\binom{i}{j}d_{n-j}(x)\quad\text{for any } 0\leqslant i\leqslant n.$$

In particular, $d_{n,1}^B(x)=d_{n,0}^B(x)+d_{n-1,0}^B(x)=d_n(x)+d_{n-1}(x)$.

For $1\leqslant i\leqslant n$, let $\overline{\mathcal{D}}_{n,i}^B$ be the set of type B derangements in $\mathcal{D}_n^B$ with the restriction that the set of negative entries of each derangement is $\{\overline{j_1},\overline{j_2},\ldots,\overline{j_i}\}$, where $\{j_1,j_2,\ldots,j_i\}$ is a subset of $[n]$ with i elements and $\overline{j_\ell}=-j_\ell$ for $\ell\in[i]$. Let $\overline{d}_{n,i}^B(x)$ be the derangement polynomials over $\overline{\mathcal{D}}_{n,i}^B$. In other words,

$$\overline{d}_{n,i}^B(x)=\sum_{\sigma\in\overline{\mathcal{D}}_{n,i}^B}x^{\operatorname{exc}(\sigma)}.$$

There is an order-preserving bijection $\phi_4:\overline{\mathcal{D}}_{n,i}^B\mapsto\widetilde{\mathcal{D}}_{n,i}^B$. Define

$$\mathcal{C}(\sigma)=\{\overline{j_1},\overline{j_2},\ldots,\overline{j_i}\}\cup([n]\setminus\{j_1,j_2,\ldots,j_i\}),$$
$$\mathcal{D}(\sigma)=\{\overline{n},\overline{n-1},\ldots,\overline{n-i+1}\}\cup\{1,2,\ldots,n-i\}.$$

We write the elements in $\mathcal{C}(\sigma)$ and $\mathcal{D}(\sigma)$ in increasing order. If $\sigma(j)$ is the kth element of $\mathcal{C}(\sigma)$, then let $\phi_4(\sigma)(j)$ be the kth element of $\mathcal{D}(\sigma)$. See Example 5.12 for an illustration. Along the same lines, one can define the inverse of ϕ_4. Clearly, the order-preserving bijection ϕ_4 does not change the number of excedances. Thus $\overline{d}_{n,i}^B(x)=d_{n,i}^B(x)$, which yields that

$$d_n^B(x,p)=\sum_{i=0}^{n}\binom{n}{i}d_{n,i}^B(x)p^i=\sum_{i=0}^{n}\binom{n}{i}p^i\sum_{j=0}^{i}\binom{i}{j}d_{n-j}(x),$$

where the first equality follows by choosing i negative entries. □

Example 5.12. Let $\sigma=(1,4,\overline{3},9,8)(2,\overline{5})(\overline{6})(\overline{7})\in\overline{\mathcal{D}}_{9,4}^B$. We have

$$\mathcal{C}(\sigma)=\{\overline{7},\overline{6},\overline{5},\overline{3},1,2,4,8,9\},$$

$$\mathcal{D}(\sigma)=\{\overline{9},\overline{8},\overline{7},\overline{6},1,2,3,4,5\}.$$

The order-preserving bijection between $\mathcal{A}(\sigma)$ and $\mathcal{B}(\sigma)$ can be illustrated by the following array:

$$\begin{pmatrix}\overline{7}&\overline{6}&\overline{5}&\overline{3}&1&2&4&8&9\\\overline{9}&\overline{8}&\overline{7}&\overline{6}&1&2&3&4&5\end{pmatrix}.$$

Thus $\phi_4(\sigma)=(1,3,\overline{6},5,4)(2,\overline{7})(\overline{8})(\overline{9})$.

The *type D Coxeter group* $\mathcal{S}_n^D$ is the subgroup of $\mathcal{S}_n^B$ consisting of signed permutations with an even number of negative entries. Let $\mathcal{D}_n^D=\{\sigma\in\mathcal{S}_n^D:\operatorname{fix}(\sigma)=0\}$ be the set of all derangements in

$\mathcal{S}_n^D$. The *derangement polynomials of type D* are defined by

$$d_n^D(x)=\sum_{\sigma\in\mathcal{D}_n^D}x^{\operatorname{exc}(\sigma)}.$$

Using Theorem 5.11, we get

$$d_n^B(x,p)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}p^{2i}\sum_{j=0}^{2i}\binom{2i}{j}d_{n-j}(x)+\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\binom{n}{2i+1}p^{2i+1}\sum_{j=0}^{2i+1}\binom{2i+1}{j}d_{n-j}(x).$$

Since the parameter p marks negative entries, we get the following result.

Theorem 5.13. *For any $n\geqslant 1$, one has*

$$d_n^D(x)=\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}\sum_{j=0}^{2i}\binom{2i}{j}d_{n-j}(x).$$

6. The flag excedance statistic of signed permutations

We say that $i\in[n]$ is an index of *type A excedance* if $\sigma(i)>i$. For $\sigma\in\mathcal{S}_n^B$, we let

$$\begin{aligned}&\operatorname{exc}_A(\sigma)=\#\{i\in[n]:\sigma(i)>i\},\ \operatorname{neg}(\sigma)=\#\{|\sigma(i)|\in[n]:\sigma(i)<0\},\\&\operatorname{fix}(\sigma)=\#\{i\in[n]:\sigma(i)=i\},\ \operatorname{single}(\sigma)=\#\{i\in[n]:\sigma(i)=\bar{i}\},\\&\operatorname{fexc}(\sigma)=2\operatorname{exc}_A(\sigma)+\operatorname{neg}(\sigma),\ \ \operatorname{aexc}_A(\sigma)=n-\operatorname{exc}_A(\sigma)-\operatorname{fix}(\sigma)-\operatorname{single}(\sigma).\end{aligned}$$

Example 6.1. The cycle decomposition of $\sigma=2\,\bar{5}\,1\,3\,4\,\bar{6}\,8\,7\,9$ is $(1,2,\bar{5},4,3)(\bar{6})(7,8)(9)$. Moreover,

$$\operatorname{exc}_A(\sigma)=\#\{1,7\}=2,\ \operatorname{neg}(\sigma)=\#\{5,6\}=2,\ \operatorname{fexc}(\sigma)=6,\ \operatorname{single}(\sigma)=\operatorname{fix}(\sigma)=1.$$

The flag excedance statistic fexc has been extensively studied by Bagno–Garber [5], Foata–Han [25], Mongelli [45], Shin–Zeng [51] and Zhuang [55]. In particular, Mongelli [45, Section 3] found the following two formulas:

$$\sum_{\sigma\in\mathcal{S}_n^B}x^{2\operatorname{exc}_A(\sigma)}p^{\operatorname{neg}(\sigma)}=(1+p)^nA_n\left(\frac{x^2+p}{1+p}\right),\tag{48}$$

$$\sum_{\sigma\in\mathcal{D}_n^B}x^{2\operatorname{exc}_A(\sigma)}p^{\operatorname{neg}(\sigma)}=\sum_{k=0}^{n}\binom{n}{k}(1+p)^kp^{n-k}d_k\left(\frac{x^2+p}{1+p}\right).\tag{49}$$

Setting $p=x$ in (48) leads to a classical formula (see [1,25] for instance):

$$\sum_{\sigma\in\mathcal{S}_n^B}x^{\operatorname{fexc}(\sigma)}=(1+x)^nA_n(x).$$

The *flag derangement polynomials* are defined by

$$D_n^B(x)=\sum_{\sigma\in\mathcal{D}_n^B}x^{\operatorname{fexc}(\sigma)}.$$

It follows from (49) that

$$D_n^B(x)=\sum_{\sigma\in\mathcal{D}_n^B}x^{2\operatorname{exc}_A(\sigma)}x^{\operatorname{neg}(\sigma)}=\sum_{k=0}^{n}\binom{n}{k}(1+x)^kx^{n-k}d_k(x).$$

Using the above expansion, Mongelli [45, Proposition 3.5] proved the symmetry of $D_n^B(x)$. Subsequently, Shin–Zeng [51, Corollary 5] proved that the polynomials $D_n^B(x)$ are γ-positive.

Let

$$B_n^{(A)}(x,y,s,t,p,q)=\sum_{\sigma\in\mathcal{S}_n^B}x^{\mathrm{exc}_A(\sigma)}y^{\mathrm{aexc}_A(\sigma)}s^{\mathrm{single}\,(\sigma)}t^{\mathrm{fix}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)}.$$

Now we give the main result of this section.

Theorem 6.2. *One has*

$$B_n^{(A)}(x,y,s,t,p,q)=\sum_{\pi\in\mathcal{S}_n}(x+py)^{\mathrm{exc}\,(\pi)}(y+py)^{\mathrm{drop}\,(\pi)}(t+sp)^{\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\sigma)}. \tag{50}$$

Proof. We claim that if

$$G_6=\{I\to Iq(t+sp),\ t\to y(x+py),\ s\to y(x+py),\ x\to y(x+py),\ y\to y(x+py)\},$$

then

$$D_{G_6}^n(I)=I\sum_{\sigma\in\mathcal{S}_n^B}x^{\mathrm{exc}_A(\sigma)}y^{\mathrm{aexc}_A(\sigma)}s^{\mathrm{single}\,(\sigma)}t^{\mathrm{fix}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)}. \tag{51}$$

Let $\sigma\in\mathcal{S}_n^B$. We introduce a grammatical labeling of σ as follows:

(L_1) put a subscript label p right after every negative element of σ;
(L_2) if i is a fixed point, then put a superscript label t right after i, i.e., (i^t);
(L_3) if i is a singleton, then put a superscript label s right after $\bar{i}$, i.e., $(\bar{i}^s)$;
(L_4) if $\sigma(i)>i$, then put a superscript label x just before $\sigma(i)$;
(L_5) if $\sigma(i)<i$, then put a superscript label y right after i or $\bar{i}$;
(L_6) put a subscript label q right after each cycle;
(L_7) put a superscript label I right after σ.

For example, if $\sigma=(1,2,\overline{5},4,3)(\overline{6})(7,8)(9,\overline{10})(\overline{11},12)(13)$, then the grammatical labeling of σ is given as follows:

$$(1^x2^y\overline{5}_p^y4^y3^y)_q(\overline{6}_p^s)_q(7^x8^y)_q(9^y\overline{10}_p^y)_q(\overline{11}_p^x12^y)_q(13^t)_q^I.$$

The weight of σ is given by $Ix^{\mathrm{exc}_A(\sigma)}y^{\mathrm{aexc}_A(\sigma)}s^{\mathrm{single}\,(\sigma)}t^{\mathrm{fix}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)}$. Every permutation in $\mathcal{S}_n^B$ can be obtained from a permutation in $\mathcal{S}_{n-1}^B$ by inserting n or $\overline{n}$. For $n=1$, we have $\mathcal{S}_1^B=\{(1^t)_q^I,\ (\overline{1}_p^s)_q^I\}$. Note that $D_{G_6}(I)=Iq(t+sp)$. Then the sum of weights of the elements in $\mathcal{S}_1^B$ is given by $D_{G_6}(I)$. Hence the claim holds for $n=1$.

We proceed by induction on n. Suppose that we get all labeled permutations in $\mathcal{S}_{n-1}^B$, where $n\geqslant 2$. Let $\widetilde{\sigma}$ be obtained from $\sigma\in\mathcal{S}_{n-1}^B$ by inserting n or $\overline{n}$. When the inserted n or $\overline{n}$ forms a new cycle, the insertion corresponds to the substitution rule $I\to Iq(t+sp)$. Now we insert n or $\overline{n}$ right after $\sigma(i)$. If i is a fixed point or a singleton of σ, then the changes of labeling are respectively illustrated as follows:

$$\cdots(i^t)_q\cdots\to\cdots(i^xn^y)_q\cdots,\ \cdots(i^t)_q\cdots\to\cdots(i^y\overline{n}_p^y)_q\cdots,$$

$$\cdots(\overline{i}_p^s)_q\cdots\to\cdots(\overline{i}_p^xn^y)_q\cdots,\ \cdots(\overline{i}_p^s)_q\cdots\to\cdots(\overline{i}_p^y\overline{n}_p^y)_q\cdots.$$

If i is an excedance of type A, then the changes of labeling are respectively illustrated as follows:

$$\cdots(\cdots\sigma(i)^x\sigma(|\sigma(i)|)\cdots)_q\cdots\to\cdots(\cdots\sigma(i)^xn^y\sigma(|\sigma(i)|)\cdots)_q\cdots,$$

$$\cdots(\cdots\sigma(i)^x\sigma(|\sigma(i)|)\cdots)_q\cdots\to\cdots(\cdots\sigma(i)^y\overline{n}_p^y\sigma(|\sigma(i)|)\cdots)_q\cdots.$$

The same argument applies to the case when the new element is inserted right after an element labeled by y. In each case, the insertion of n or $\overline{n}$ corresponds to one substitution rule in G_6.

Therefore, the action of D_{G_6} on the set of weighted signed permutations in $\mathcal{S}_{n-1}^B$ gives the set of weighted signed permutations in $\mathcal{S}_n^B$, and so (51) holds.

For the grammar G_6, setting $A = t + sp$, $B = x + py$ and $C = (1 + p)y$, we get

$$D_{G_6}(I) = IqA,\ D_{G_6}(A) = BC,\ D_{G_6}(B) = BC,\ D_{G_6}(C) = BC.$$

Let $G_7 = \{I \to IqA, A \to BC, B \to BC, C \to BC\}$. It follows from Lemma 3.12 that

$$D_{G_7}^n(I) = I \sum_{\pi\in\mathcal{S}_n} A^{\mathrm{fix}\,(\pi)}B^{\mathrm{exc}\,(\pi)}C^{\mathrm{drop}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}. \tag{52}$$

Then upon substituting $A = t + sp$, $B = x + py$ and $C = (1 + p)y$ in (52), we get (50). □

Comparing Proposition 3.11 with Theorem 6.2, we get

$$B_n^{(A)}(x, y, s, t, p, q) = (1 + p)^n y^n A_n\left(\frac{x + py}{y + py}, \frac{t + sp}{y + py}, q\right). \tag{53}$$

It follows from (10) and (53) that

$$\sum_{n=0}^{\infty} B_n^{(A)}(x, y, s, t, p, q)\frac{z^n}{n!} = \left(\frac{(y - x)e^{(t+sp)z}}{(1 + p)ye^{(x+py)z} - (x + py)e^{(1+p)yz}}\right)^q. \tag{54}$$

Note that

$$B_n^{(A)}(x^2, 1, 1, 0, px, q) = \sum_{\sigma\in\mathcal{D}_n^B} x^{\mathrm{fexc}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)}.$$

It follows from (54) that

$$\sum_{n=0}^{\infty} B_n^{(A)}(x^2, 1, 1, 0, px, q)\frac{z^n}{n!} = \left(\frac{(1 - x^2)e^{pxz}}{(1 + px)e^{(x^2+px)z} - (x^2 + px)e^{(1+px)z}}\right)^q.$$

In the sequel, we consider the γ-positivity of the following polynomials:

$$F_n^{(\mathrm{fexc}\,,\mathrm{cyc}\,)}(x, q) = \sum_{\sigma\in\mathcal{S}_n^B} x^{\mathrm{fexc}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)},$$

$$D_n^{(\mathrm{fexc}\,,\mathrm{cyc}\,)}(x, q) = \sum_{\sigma\in\mathcal{D}_n^B} x^{\mathrm{fexc}\,(\sigma)}q^{\mathrm{cyc}\,(\sigma)},$$

$$E_n^{(\mathrm{fexc}\,,\mathrm{neg}\,)}(x, p) = \sum_{\sigma\in\mathcal{S}_n^B} x^{\mathrm{fexc}\,(\sigma)}p^{\mathrm{neg}\,(\sigma)}.$$

From (53), we see that

$$F_n^{(\mathrm{fexc}\,,\mathrm{cyc}\,)}(x, q) = B_n^{(A)}(x^2, 1, 1, 1, x, q) = (1 + x)^n A_n(x, q) = (1 + x)^n \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{exc}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

Using Lemma 2.2 and Theorem 3.6, we get the following corollary.

Corollary 6.3. *Let $0 \leqslant q \leqslant 1$ be a given real number. Then $F_n^{(\mathrm{fexc}\,,\mathrm{cyc}\,)}(x, q)$ are bi-γ-positive.*

Using (53), we get

$$D_n^{(\mathrm{fexc}\,,\mathrm{cyc}\,)}(x, q) = B_n^{(A)}(x^2, 1, 1, 0, x, q) = (1 + x)^n A_n\left(x, \frac{x}{1 + x}, q\right).$$

Combining this with (2), we obtain

$$D_n^{(\mathrm{fexc}\,,\mathrm{cyc}\,)}(x, q) = \sum_{\pi\in\mathcal{S}_n} x^{\mathrm{exc}\,(\pi)}x^{\mathrm{fix}\,(\pi)}(1 + x)^{n-\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (qx)^i (1+x)^{n-i} \sum_{\pi \in \mathcal{D}_{n-i}} x^{\mathrm{exc}(\pi)} q^{\mathrm{cyc}(\pi)},$$

where the second equality follows by choosing i fixed points. Note that

$$(qx)^i (1+x)^{n-i} \sum_{\pi \in \mathcal{D}_{n-i}} x^{\mathrm{exc}(\pi)} q^{\mathrm{cyc}(\pi)} = (qx)^i (1+x)^{n-i} d_{n-i}(x,q).$$

Let $q > 0$ be a given real number. Then $(qx)^i(1+x)^{n-i}$ are γ-positive polynomials for all $0 \leqslant i \leqslant n$. From Proposition 3.2, we see that $d_{n-i}(x,q)$ are also γ-positive. By Lemma 2.2, we find that the polynomials $(qx)^i(1+x)^{n-i}d_{n-i}(x,q)$ are all γ-positive with the same center of symmetry. So we get the following result, which is a generalization of a result of Shin–Zeng [51, Corollary 5].

Corollary 6.4. *Let $q > 0$ be a given real number. Then $D_n^{(\mathrm{fexc},\mathrm{cyc})}(x,q)$ are γ-positive.*

Using (53), we get

$$E_n^{(\mathrm{fexc},\mathrm{neg})}(x,p) = B_n^{(A)}(x^2,1,1,1,px,1) = (1+px)^n A_n\left(\frac{x^2+px}{1+px}\right). \tag{55}$$

In particular, $E_n^{(\mathrm{fexc},\mathrm{neg})}(x,-1) = (1-x)^n A_n(-x)$. We can now present the following result.

Theorem 6.5. *Let $p \geqslant 1$ be a given real number. Then $E_n^{(\mathrm{fexc},\mathrm{neg})}(x,p)$ are bi-γ-positive.*

Proof. For $n \geqslant 1$, combining (55) and Proposition 3.1, we obtain

$$E_n^{(\mathrm{fexc},\mathrm{neg})}(x,p) = (1+xp)\widetilde{E}_n(x,p) = (1+x)\widetilde{E}_n(x,p) + (p-1)x\widetilde{E}_n(x,p),$$

where

$$\widetilde{E}_n(x,p) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i}(x^2+px)^i(1+px)^i(1+2px+x^2)^{n-1-2i}.$$

Note that $(x^2+px)(1+px) = x(p(1+x)^2+(p-1)^2x)$, $1+2px+x^2 = (1+x)^2+2(p-1)x$. Therefore, we get

$$\widetilde{E}_n(x,p) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,i}x^i[p(1+x)^2+(p-1)^2x]^i[(1+x)^2+2(p-1)x]^{n-1-2i},$$

where $\gamma_{n,i} \geqslant 0$ for all $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$ and $p \geqslant 1$. By using Lemma 2.2, we find that for $0 \leqslant i \leqslant \lfloor (n-1)/2 \rfloor$, the polynomials $\gamma_{n,i}x^i[p(1+x)^2+(p-1)^2x]^i[(1+x)^2+2(p-1)x]^{n-1-2i}$ are all γ-positive with the same center of symmetry, and so $\widetilde{E}_n(x,p)$ is γ-positive, which yields the desired result. □

7. Excedance-type polynomials of colored permutations

7.1. Preliminary

Let r be a fixed positive integer. An *r-colored permutation* of length n can be written as π^c, where $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{S}_n$ and $c = (c_1,c_2,\ldots,c_n) \in [0,r-1]^n$, i.e., c_i is a nonnegative integer in the interval $[0,r-1]$ for any $i \in [n]$. As usual, π^c can be denoted as $\pi_1^{c_1}\pi_2^{c_2}\cdots\pi_n^{c_n}$, where c_i can be thought of as the color assigned to π_i. Denote by $\mathbb{Z}_r \wr \mathcal{S}_n$ the set of all r-colored permutations of length n. The wreath product $\mathbb{Z}_r \wr \mathcal{S}_n$ could be considered as the colored permutation group $G_{r,n}$ consists of all permutations of the alphabet Σ of rn letters:

$$\Sigma = \{1,2,\ldots,n,\overline{1},\ldots,\overline{n},\ldots,1^{[r-1]},\ldots,n^{[r-1]}\}$$

satisfying $\pi(\bar{i}) = \overline{\pi(i)}$. In particular, $\mathbb{Z}_1 \wr \mathcal{S}_n = \mathcal{S}_n$ and $\mathbb{Z}_2 \wr \mathcal{S}_n = \mathcal{S}_n^B$. Following Steingrímsson [54], for $1 \leqslant i \leqslant n$, an index i is an *excedance* of π^c if $i <_f \pi_i$, where we use the order $<_f$ of Σ:

$$1 <_f \bar{1} <_f \cdots <_f 1^{[r-1]} <_f 2 <_f \bar{2} <_f \cdots <_f 2^{[r-1]} <_f \cdots <_f n <_f \bar{n} <_f \cdots <_f n^{[r-1]}. \tag{56}$$

Let $\mathrm{exc}\,(\pi^c)$ be the number of excedances of π^c. A *fixed point* of $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n$ is an entry $\pi_k^{c_k}$ such that $\pi_k = k$ and $c_k = 0$. An element $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n$ is called a *derangement* if it has no fixed points. Let $\mathfrak{D}_{n,r}$ be the set of derangements in $\mathbb{Z}_r \wr \mathcal{S}_n$. The *q-colored derangement polynomials* are defined by

$$d_{n,r}(x, q) = \sum_{\pi^c \in \mathfrak{D}_{n,r}} x^{\mathrm{exc}\,(\pi^c)} q^{\mathrm{cyc}\,(\pi^c)}.$$

Let $d_{n,r}(x) := d_{n,r}(x, 1)$ be the *colored derangement polynomials*. According to [23, Theorem 5],

$$\sum_{n=0}^{\infty} d_{n,r}(x) \frac{z^n}{n!} = \frac{(1-x)e^{(r-1)xz}}{e^{rxz} - xe^{rz}}.$$

There has been much work on the polynomials $d_{n,r}(x)$, see [23,29,51] for instance. By using the theory of Rees products of posets, Athanasiadis [2, Theorem 1.3] obtained the following result.

Theorem 7.1 (*[2, Theorem 1.3]*). *We have $d_{n,r}(x) = d^+_{n,r}(x) + d^-_{n,r}(x)$, where*

$$d^+_{n,r}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \beta^+_{n,r,i} x^i (1+x)^{n-2i}, \quad d^-_{n,r}(x) = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \beta^-_{n,r,i} x^i (1+x)^{n+1-2i}.$$

The reader is referred to [2, Theorem 1.3], [29, Theorem 4.6] and [32, Proposition 1.5] for various combinatorial interpretations of $d^+_{n,r}(x)$ and $d^-_{n,r}(x)$. Recently, by defining the modified Foata–Strehl action on colored permutations, Han [32] gave a combinatorial proof of [2, Theorem 1.3]. Moreover, Brändén–Solus [10, Corollary 3.9] and Gustafsson–Solus [29, Theorem 5.1] showed that both $d^+_{n,r}(x)$ and $d^-_{n,r}(x)$ have only real zeros.

Following Steingrímsson [54], the *r-colored Eulerian polynomials* are defined by

$$A_{n,r}(x) = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{exc}\,(\pi^c)},$$

which satisfy the recurrence relation

$$A_{n,r}(x) = (1 + (rn-1)x)A_{n-1,r}(x) + rx(1-x)\frac{\mathrm{d}}{\mathrm{d}x}A_{n,r}(x), \ A_{0,r}(x) = 1. \tag{57}$$

Let $A_{n,r}(x) = \sum_{k=0}^{n} A_r(n,k)x^k$. Equating the coefficient of x^k in both sides of (57), one can derive that

$$A_r(n,k) = (rk+1)A_r(n-1,k) + (r(n-k)+r-1)A_r(n-1,k-1), \tag{58}$$

with $A_r(0,k) = \delta_{0,k}$ (see [54, Lemma 16]). According to [54, Theorem 20], we have

$$\sum_{n=0}^{\infty} A_{n,r}(x) \frac{z^n}{n!} = \frac{(1-x)e^{z(1-x)}}{1 - xe^{rz(1-x)}}. \tag{59}$$

When $r = 1$ and $r = 2$, the polynomial $A_{n,r}(x)$ reduces to the types A and B Eulerian polynomials $A_n(x)$ and $B_n(x)$, respectively. Comparing (10) with (59), one can see that

$$A_{n,r}(x) = r^n A_n\left(x, \frac{1+(r-1)x}{r}, 1\right),$$

since it follows from (10) that

$$\sum_{n=0}^{\infty} r^n A_n\left(x, \frac{1+(r-1)x}{r}, 1\right) \frac{z^n}{n!} = \frac{(1-x)e^{z(1-x)}e^{rxz}}{e^{rxz} - xe^{rz}} = \frac{(1-x)e^{z(1-x)}}{1 - xe^{rz(1-x)}}.$$

Following [29, Corollary 4.4], we have

$$d_{n,r}(x)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}A_{i,r(x)}.$$

Recently, Athanasiadis [4, Eq. (21)] considered the following expansion:

$$A_{n,r}(x)=A^{+}_{n,r}(x)+A^{-}_{n,r}(x), \tag{60}$$

where

$$A^{+}_{n,r}(x)=\sum_{k=0}^{n}\binom{n}{k}d^{+}_{k,r}(x),\ \ A^{-}_{n,r}(x)=\sum_{k=0}^{n}\binom{n}{k}d^{-}_{k,r}(x).$$

7.2. An equivalent result to Theorem 3.6

In [23, Proposition 4], Chow and Mansour found that

$$d_{n,r}(x)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{wexc}\,(\pi)}(r-1)^{\mathrm{fix}\,(\pi)}r^{n-\mathrm{fix}\,(\pi)},$$

where $\mathrm{wexc}\,(\pi)=\#\{i\in[n]:\pi(i)\geqslant i\}$ is the number of *weak excedances* of π. Note that $\mathrm{wexc}\,(\pi)=\mathrm{exc}\,(\pi)+\mathrm{fix}\,(\pi)$. Along the same lines of its proof of [23, Proposition 4], one can derive that

$$d_{n,r}(x,q)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{wexc}\,(\pi)}(r-1)^{\mathrm{fix}\,(\pi)}r^{n-\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}. \tag{61}$$

Define

$$a_n(x,p,q)=A_n(x,xp,q)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{wexc}\,(\pi)}p^{\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

In particular,

$$a_n(x,1,q)=\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{wexc}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

Let π^{-1} be the inverse of π. The bijection $\pi\to\pi^{-1}$ on $\mathcal{S}_n$ shows that $(\mathrm{exc}\,,\mathrm{fix}\,,\mathrm{cyc}\,)$ is equidistributed with $(\mathrm{drop}\,,\mathrm{fix}\,,\mathrm{cyc}\,)$. Thus $(\mathrm{wexc}\,,\mathrm{fix}\,,\mathrm{cyc}\,)$ is equidistributed with $(n-\mathrm{exc}\,,\mathrm{fix}\,,\mathrm{cyc}\,)$. Therefore, we get

$$a_n(x,p,q)=x^nA_n\left(\frac{1}{x},p,q\right).$$

It should be noted that if $p>0$ and $q>0$, then $\deg a_n(x,p,q)=n$. Moreover, $a_n(0,p,q)=0$ and the coefficient of the highest degree term of $a_n(x,p,q)$ is p^nq^n, which corresponds to the identity permutation $12\cdots n$. So we have $a_n(0,p,q)<p^nq^n$ when $p>0$ and $q>0$. Therefore, an equivalent result to Theorem 3.6 is given as follows.

Theorem 7.2. *Let $p\in[0,1]$ and $q\in[0,1]$ be two given real numbers. The polynomials $a_n(x,1,q)$ are bi-γ-positive and the polynomials $a_n(x,p,q)$ are alternatingly increasing for $n\geqslant 1$.*

By (61), we see that

$$d_{n,r}(x,q)=r^na_n\left(x,\frac{r-1}{r},q\right).$$

So a special case of Theorem 7.2 is given as follows.

Corollary 7.3. *Let $q\in[0,1]$ be a given real number and let r be a fixed positive integer. Then the polynomials $d_{n,r}(x,q)$ are alternatingly increasing for $n\geqslant 1$.*

7.3. The bi-γ-positivity of colored Eulerian polynomials

In order to deduce the recurrence system of the bi-γ-coefficients of $A_{n,r}(x)$, we now give a grammatical description of $A_r(n,k)$, which gives an extension of Proposition 3.10.

Lemma 7.4. *If $G_8 = \{u \to uv^r, v \to u^r v\}$, then we have*

$$D^n_{G_8}(u^{r-1}v) = u^{r-1}v\sum_{k=0}^{n} A_r(n,k)u^{(n-k)r}v^{kr}. \tag{62}$$

Proof. Note that $D^0_{G_8}(u^{r-1}v) = u^{r-1}v$ and $D_{G_8}(u^{r-1}v) = u^{r-1}v(u^r+(r-1)v^r)$. Assume that the result holds for $n = m$, where $m \geqslant 1$. Then

$$\begin{aligned} &D^{m+1}_{G_8}(u^{r-1}v)\\ &= D_{G_8}\left(D^m_{G_8}(u^{r-1}v)\right)\\ &= D_{G_8}\left(u^{r-1}v\sum_{k=0}^{m} A_r(m,k)u^{(m-k)r}v^{kr}\right)\\ &= u^{r-1}v\sum_{k=0}^{m} A_r(m,k)((mr-kr+r-1)u^{(m-k)r}v^{(k+1)r} + (kr+1)u^{(m-k+1)r}v^{kr}). \end{aligned}$$

Extracting the coefficient $u^{r-1}vu^{(m-k+1)r}v^{kr}$ in the last expression, we get

$$(rk+1)A_r(m,k) + (r(m+1-k)+r-1)A_r(m,k-1).$$

Comparing the above expression of coefficients with (58), we see that $D^{m+1}_{G_8}(u^{r-1}v)$ can be written as follows:

$$D^{m+1}_{G_8}(u^{r-1}v) = u^{r-1}v\sum_{k=0}^{m+1} A_r(m+1,k)u^{(m-k+1)r}v^{kr}.$$

Hence the result holds for $n = m+1$. This completes the proof. □

Recall that $A_{n,1}(x) = A_n(x)$ and $A_{n,2}(x) = B_n(x)$, which are both γ-positive polynomials. We can now present the following result.

Theorem 7.5. *Let $r \geqslant 2$ be a given real number. For $n \geqslant 1$, we have*

$$A_{n,r}(x) = \sum_{k=0}^{\lfloor n/2\rfloor} \alpha^+_{n,k;r}x^k(1+x)^{n-2k} + x\sum_{k=0}^{\lfloor (n-1)/2\rfloor} \alpha^-_{n,k;r}x^k(1+x)^{n-1-2k}, \tag{63}$$

where the coefficients $\alpha^+_{n,k;r}$ and $\alpha^-_{n,k;r}$ satisfy the recurrence system

$$\begin{cases} \alpha^+_{n+1,k;r} = (1+rk)\alpha^+_{n,k;r} + 2r(n-2k+2)\alpha^+_{n,k-1;r} + 2\alpha^-_{n,k-1;r},\\ \alpha^-_{n+1,k;r} = (r-2)\alpha^+_{n,k;r} + (r-1+rk)\alpha^-_{n,k;r} + 2r(n-2k+1)\alpha^-_{n,k-1;r}, \end{cases}$$

with the initial conditions $\alpha^+_{1,0;r} = 1$, $\alpha^-_{1,0;r} = r-2$, $\alpha^+_{1,k;r} = \alpha^-_{1,k;r} = 0$ for $k \neq 0$. So the polynomials $A_{n,r}(x)$ are bi-γ-positive.

Proof. Consider a change of the grammar given in Lemma 7.4. Note that

$$\begin{aligned} D_{G_8}(u^rv^r) &= ru^rv^r(u^r+v^r),\ D_{G_8}(u^r+v^r) = 2ru^rv^r,\\ D_{G_8}(u^{r-1}v) &= (r-1)u^{r-1}v^{r+1} + u^{2r-1}v = (r-2)u^{r-1}v^{r+1} + u^{r-1}v(u^r+v^r),\\ D_{G_8}(u^{r-1}v^{r+1}) &= (r-1)u^{r-1}v^{r+1}(u^r+v^r) + 2u^{r-1}v(u^rv^r). \end{aligned}$$

Setting $a = u^r v^r$, $b = u^r + v^r$, $c = u^{r-1}v^{r+1}$ and $I = u^{r-1}v$, we obtain

$$D_{G_8}(a) = rab,\ D_{G_8}(b) = 2ra,$$

$$D_{G_8}(c) = (r-1)bc + 2Ia,\ D_{G_8}(I) = (r-2)c + Ib.$$

It should be noted that

$$c = Iv^r. \tag{64}$$

Consider the grammar

$$G_9 = \{I \to Ib + (r-2)c,\ a \to rab,\ b \to 2ra,\ c \to (r-1)bc + 2Ia\}.$$

By induction, it is routine to check that there exist nonnegative integers such that

$$D_{G_9}^n(I) = \sum_{k=0}^{\lfloor n/2 \rfloor} \alpha_{n,k;r}^{+} a^k b^{n-2k} I + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \alpha_{n,k;r}^{-} a^k b^{n-1-2k} c. \tag{65}$$

In particular, $D_{G_9}(I) = Ib + (r-2)c$, $D_{G_9}^2(I) = (4(r-1)a + b^2)I + r(r-2)bc$. We proceed to the inductive step. Note that

$$D_{G_9}^{n+1}(I) = \sum_k \alpha_{n,k;r}^{+} b^{n-1-2k}((1+rk)a^k b^2 I + 2r(n-2k)a^{k+1}I + (r-2)a^k bc) +$$
$$\sum_k \alpha_{n,k;r}^{-} b^{n-2-2k}((r-1+rk)a^k b^2 c + 2r(n-1-2k)a^{k+1}c + 2a^{k+1}bI).$$

Extracting coefficients of $a^k b^{n+1-2k} I$ and $a^k b^{n-2k} c$ on both sides and simplifying yields the desired recurrence system. Setting $u^r = 1$ and $v^r = x$, we have $a = x$, $b = 1 + x$. Moreover, it follows from (64) that $c = Ix$. Comparing (62) with (65), we get (63). □

It should be noted that the bi-γ-positivity of $A_{n,r}(x)$ follows already from the result of Branden–Solus [10, Theorem 3.1] that these polynomials have real-rooted symmetric decompositions. The recurrence system of the bi-γ-coefficients of $A_{n,r}(x)$ is deduced by the change of grammar method. It would be interesting to further study this recurrence system.

In the sequel, we give an application of Theorem 7.5. Following Brenti [13, Eq. (10)], the *q-Eulerian polynomials type B* are defined by

$$B_n(x, q) = \sum_{\pi \in \mathcal{S}_n^B} x^{\mathrm{des}_B(\pi)} q^{\mathrm{neg}(\sigma)}.$$

When $q = 1$, the polynomial $B_n(x, q)$ is reduced to $B_n(x)$. The polynomials $B_n(x, q)$ satisfy the recurrence relation

$$B_n(x, q) = (1 + (1+q)nx - x)B_{n-1}(x, q) + (1+q)(x - x^2)\frac{\partial}{\partial x}B_{n-1}(x, q),\ B_0(x, q) = 1.$$

The exponential generating function of $B_n(x, q)$ is given as follows (see [13, Theorem 3.4]):

$$\sum_{n=0}^{\infty} B_n(x, q)\frac{z^n}{n!} = \frac{(1-x)e^{z(1-x)}}{1 - xe^{z(1-x)(1+q)}}. \tag{66}$$

Comparing (59) with (66), we see that $A_{n,q+1}(x) = B_n(x, q)$. As a special case of Theorem 7.5, we have the following result.

Corollary 7.6. *Let $q \geqslant 1$ be a given real number. Then $B_n(x, q)$ are bi-γ-positive.*

7.4. Two multivariate colored Eulerian polynomials

Let $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n$. Recall that $\mathrm{exc}(\pi^c) = \#\{i \in [n] : i <_f \pi_i\}$. We define

$$\mathrm{aexc}(\pi^c) = \#\{i \in [n] : \pi_i <_f i\},\ \ \mathrm{fix}(\pi^c) = \#\{i \in [n] : \pi_i = i \text{ and } c_i = 0\}.$$

Consider the following *multivariate colored Eulerian polynomials*:

$$A_{n,r}(x, y, p, q) = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{exc}(\pi^c)} y^{\mathrm{aexc}(\pi^c)} p^{\mathrm{fix}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}.$$

Clearly, $A_{n,1}(x, 1, p, q) = A_n(x, p, q)$, $A_{n,r}(x, 1, 1, 1) = A_{n,r}(x)$ and $A_{n,r}(x, 1, 0, q) = d_{n,r}(x, q)$.

Lemma 7.7. *If $G_{10} = \{I \to qI\left((r-1)x + p\right), x \to rxy, y \to rxy, p \to rxy\}$, then*

$$D^n_{G_{10}}(I) = I \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{exc}(\pi^c)} y^{\mathrm{aexc}(\pi^c)} p^{\mathrm{fix}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}. \tag{67}$$

Proof. We now introduce a grammatical labeling of $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n$ as follows:

(L_1) if $i <_f \pi_i$, then we label $\pi_i^{c_i}$ by a subscript label x;
(L_2) if $\pi_i <_f i$, then we label $\pi_i^{c_i}$ by a subscript label y;
(L_3) if $\pi_i = i$ and $c_i = 0$, then we label i by a subscript label p;
(L_4) put a subscript label I right after π^c, and put a superscript label q right after each cycle.

Note that the weight of π^c is given by

$$w(\pi^c) = Ix^{\mathrm{exc}(\pi^c)} y^{\mathrm{aexc}(\pi^c)} p^{\mathrm{fix}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}.$$

For $n = 1$, we have

$$\mathbb{Z}_r \wr \mathcal{S}_1 = \{(1_p)^q_I, (\overline{1}_x)^q_I, (1^{[2]}_x)^q_I, \ldots, (1^{[r-1]}{}_x)^q_I\}.$$

Note that $D_{G_{10}}(I) = qI((r-1)x + p)$. Then the sum of weights of the elements in $\mathbb{Z}_r \wr \mathcal{S}_1$ is given by $D_{G_{10}}(I)$. Hence the result holds for $n = 1$. We proceed by induction on n. Suppose we get all labeled permutations in $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_{n-1}$, where $n \geqslant 2$. Let $\widehat{\pi^c}$ be obtained from $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_{n-1}$ by inserting n^{c_j}, where $0 \leqslant c_j \leqslant r-1$ is a nonnegative integer. When the inserted n^{c_j} forms a new cycle, the insertion corresponds to the substitution rule $I \to qI((r-1)x + p)$ since we have r choices for c_j. For the other cases, the changes of labeling are illustrated as follows:

$$\cdots(\cdots \pi_i^{c_i}{}_x \pi_{i+1}^{c_{i+1}} \cdots)\cdots \mapsto \cdots(\cdots \pi_i^{c_i}{}_x n^{c_j}{}_y \pi_{i+1}^{c_{i+1}} \cdots)\cdots;$$

$$\cdots(\cdots \pi_i^{c_i}{}_y \pi_{i+1}^{c_{i+1}} \cdots)\cdots \mapsto \cdots(\cdots \pi_i^{c_i}{}_x n^{c_j}{}_y \pi_{i+1}^{c_{i+1}} \cdots)\cdots;$$

$$\cdots(i_p)\cdots \mapsto \cdots(i_x n^{c_j}{}_y)\cdots.$$

In each case, the insertion of n^{c_j} corresponds to one substitution rule in G_{10}. Therefore, the action of $D_{G_{10}}$ on the set of weighted colored permutations in $\mathbb{Z}_r \wr \mathcal{S}_{n-1}$ gives the set of weighted colored permutations in $\mathbb{Z}_r \wr \mathcal{S}_n$. This completes the proof. □

Now we present the following result.

Theorem 7.8. *One has*

$$A_{n,r}(x, y, p, q) = \sum_{\pi \in \mathcal{S}_n} (rx)^{\mathrm{exc}(\pi)} (ry)^{\mathrm{drop}(\pi)} ((r-1)x + p)^{\mathrm{fix}(\pi)} q^{\mathrm{cyc}(\pi)}.$$

Proof. Let G_{10} be the grammar given in Lemma 7.7. Setting $a = (r-1)x + p$, $b = rx$ and $c = ry$, we get $D_{G_{10}}(I) = qIa$, $D_{G_{10}}(a) = bc$, $D_{G_{10}}(b) = bc$, $D_{G_{10}}(c) = bc$. Let

$$G_{11} = \{I \to qIa, a \to bc, b \to bc, c \to bc\}.$$

It follows from Lemma 3.12 that

$$D^n_{G_{11}}(I) = I \sum_{\pi \in \mathcal{S}_n} b^{\mathrm{exc}(\pi)} c^{\mathrm{drop}(\pi)} a^{\mathrm{fix}(\pi)} q^{\mathrm{cyc}(\pi)}. \tag{68}$$

Then upon substituting $a = (r-1)x + p$, $b = rx$ and $c = ry$ in (68), we get the desired result. □

Comparing Proposition 3.11 with Theorem 7.8, we get

$$A_{n,r}(x, y, p, q) = (ry)^n A_n\left(\frac{x}{y}, \frac{(r-1)x+p}{ry}, q\right). \tag{69}$$

It follows from (69) that

$$A_{n,r}(x, 1, x, q) = r^n A_n(x, x, q) = r^n \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{wexc}(\pi)} q^{\mathrm{cyc}(\pi)}.$$

From Theorem 7.2, we see that the polynomials

$$A_n(x, x, q) = \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{wexc}(\pi)} q^{\mathrm{cyc}(\pi)}$$

are bi-γ-positive, where $q \in [0, 1]$ is a given real number. So we have the following result.

Corollary 7.9. *Let $q \in [0, 1]$ be a given real number. Then $A_{n,r}(x, 1, x, q)$ are bi-γ-positive.*

In the sequel, we give a unified generalization of Theorems 5.2 and 7.8. As usual, set $[0]_p = 0$. For any positive integer n, let

$$[n]_p = 1 + p + \cdots + p^{n-1}.$$

Let $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n$. A *singleton* of π^c is an index $i \in [n]$ such that $\pi_i = i$ and $c_i > 0$. We define

$$\mathrm{single}(\pi^c) = \#\{i \in [n] : \pi_i = i \text{ and } c_i > 0\},$$
$$\mathrm{exc}_B(\pi^c) = \#\{i \in [n] : i <_f \pi_i \text{ and } \pi_i \neq i\}.$$

It is clear that $\mathrm{exc}(\pi^c) = \mathrm{exc}_B(\pi^c) + \mathrm{single}(\pi^c)$. Let $\mathrm{csum}(\pi^c) = \sum_{i=1}^n c_i$. Consider the following *multivariate colored Eulerian polynomials*:

$$A_{n,r}(x, y, s, t, p, q) = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{exc}_B(\pi^c)} y^{\mathrm{aexc}(\pi^c)} s^{\mathrm{single}(\pi^c)} t^{\mathrm{fix}(\pi^c)} p^{\mathrm{csum}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}.$$

As a refinement of the grammatical labeling given in the proof of Lemma 7.7, we give another grammatical labeling of $\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n$ as follows:

(L_1) if $i <_f \pi_i$ and $\pi_i \neq i$, then we label $\pi_i^{c_i}$ by a subscript label x;
(L_2) if $\pi_i <_f i$, then we label $\pi_i^{c_i}$ by a subscript label y;
(L_3) if $\pi_i = i$ and $c_i = 0$, then we put a subscript label t just before i;
(L_4) if $\pi_i = i$ and $c_i > 0$, then we put a subscript label s just before $\pi_i^{c_i}$;
(L_5) put a subscript label p^{c_i} right after each element $\pi_i^{c_i}$ of π^c;
(L_6) put a subscript label I right after π^c, and put a superscript label q right after each cycle.

Then the weight of π^c is given by

$$w(\pi^c) = I x^{\mathrm{exc}_B(\pi^c)} y^{\mathrm{aexc}(\pi^c)} s^{\mathrm{single}(\pi^c)} t^{\mathrm{fix}(\pi^c)} p^{\mathrm{csum}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}.$$

For $n = 1$, we have $\mathbb{Z}_r \wr \mathcal{S}_1 = \{({}_t 1_{p^0})^q_I, ({}_s \overline{1}_p)^q_I, ({}_s 1^{[2]}_{p^2})^q_I, \ldots, ({}_s 1^{[r-1]}{}_{p^{r-1}})^q_I\}$. The general case follows by the same argument as the proof of Lemma 7.7 and we omit its proof for simplicity.

Lemma 7.10. *If $G_{12} = \{I \to qI\left(t + sp[r-1]_p\right), x \to [r]_p xy, y \to [r]_p xy, t \to [r]_p xy, s \to [r]_p xy\}$, then*

$$D^n_{G_{12}}(I) = I \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{exc}_B(\pi^c)} y^{\mathrm{aexc}(\pi^c)} s^{\mathrm{single}(\pi^c)} t^{\mathrm{fix}(\pi^c)} p^{\mathrm{csum}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}. \tag{70}$$

Theorem 7.11. *We have*

$$A_{n,r}(x,y,s,t,p,q)=[r]_p^n y^n A_n\left(\frac{x}{y},\frac{t+sp[r-1]_p}{[r]_p y},q\right).$$

In particular,

$$A_{n,r}(x,1,x,t,p,q)=\sum_{\pi^c\in\mathbb{Z}_r\wr\mathcal{S}_n} x^{\mathrm{exc}\,(\pi^c)}t^{\mathrm{fix}\,(\pi^c)}p^{\mathrm{csum}\,(\pi^c)}q^{\mathrm{cyc}\,(\pi^c)}=[r]_p^n A_n\left(x,\frac{t+xp[r-1]_p}{[r]_p},q\right).$$

Proof. Let G_{12} be the grammar given in Lemma 7.10. Setting $a_1=t+sp[r-1]_p,\ a_2=[r]_px,\ a_3=[r]_py$, we get

$$D_{G_{12}}(a_1)=a_2a_3,\ D_{G_{12}}(a_2)=a_2a_3,\ D_{G_{12}}(a_3)=a_2a_3.$$

Let $G_{13}=\{I\to qIa_1, a_1\to a_2a_3, a_2\to a_2a_3, a_3\to a_2a_3\}$. Then by Lemma 3.12 and Proposition 3.11,

$$D_{G_{13}}^n(I)=I\sum_{\pi\in\mathcal{S}_n} a_1^{\mathrm{fix}\,(\pi)}a_2^{\mathrm{exc}\,(\pi)}a_3^{\mathrm{drop}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}=Ia_3^nA_n\left(\frac{a_2}{a_3},\frac{a_1}{a_3},q\right).$$

Then upon substituting $a_1=t+sp[r-1]_p,\ a_2=[r]_px$ and $a_3=[r]_py$ in the above expression, we get the desired result. □

Note that $1+p[r-1]_p=[r]_p$. From Theorem 7.11, we see that

$$A_{n,r}(x,1,x,x,p,q)=[r]_p^nA_n\left(x,\frac{x+xp[r-1]_p}{[r]_p},q\right)=[r]_p^nA_n\left(x,x,q\right).$$

So we have the following result.

Corollary 7.12. *Let* $\mathrm{wexc}\,(\pi^c)=\mathrm{exc}\,(\pi^c)+\mathrm{fix}\,(\pi^c)$*. We have*

$$\sum_{\pi^c\in\mathbb{Z}_r\wr\mathcal{S}_n} x^{\mathrm{wexc}\,(\pi^c)}p^{\mathrm{csum}\,(\pi^c)}q^{\mathrm{cyc}\,(\pi^c)}=[r]_p^n\sum_{\pi\in\mathcal{S}_n}x^{\mathrm{wexc}\,(\pi)}q^{\mathrm{cyc}\,(\pi)}.$$

7.5. Another multivariate colored Eulerian polynomials

Let $\pi^c=\pi_1^{c_1}\pi_2^{c_2}\cdots\pi_n^{c_n}\in\mathbb{Z}_r\wr\mathcal{S}_n$. Following [2,4,5,51], we define

$$\mathrm{exc}_A(\pi^c)=\#\{i\in[n]:i<_c\pi_i \text{ and } c_i=0\},\quad \mathrm{aexc}_A(\pi^c)=\#\{i\in[n]:\pi_i<_ci\},$$
$$\mathrm{fix}\,(\pi^c)=\#\{i\in[n]:\pi_i=i \text{ and } c_i=0\},\ \mathrm{single}\,(\pi^c)=\#\{i\in[n]:\pi_i=i \text{ and } c_i>0\},$$
$$\mathrm{csum}\,(\pi^c)=\sum_{i=1}^n c_i,\ \ \mathrm{fexc}\,(\pi^c)=r\cdot\mathrm{exc}_A(\pi^c)+\mathrm{csum}\,(\pi^c),$$

where the comparison is with respect to the order $<_c$ of Σ:

$$1^{[r-1]}<_c2^{[r-1]}<_c\cdots<_cn^{[r-1]}<_c\cdots<_c\overline{1}<_c\overline{2}<_c\cdots<_c\overline{n}<_c1<_c2<_c\cdots<_cn.$$

Consider the following polynomials

$$A_n^{(r)}(x,y,s,t,p,q)=\sum_{\pi^c\in\mathbb{Z}_r\wr\mathcal{S}_n}x^{\mathrm{exc}_A(\pi^c)}y^{\mathrm{aexc}_A(\pi^c)}s^{\mathrm{single}\,(\pi^c)}t^{\mathrm{fix}\,(\pi^c)}p^{\mathrm{csum}\,(\pi^c)}q^{\mathrm{cyc}\,(\pi^c)}.$$

For $1\leqslant i\leqslant n$, we introduce a grammatical labeling of π^c as follows:

(L_1) put a subscript label p^{c_i} right after each element $\pi_i^{c_i}$ of π^c;
(L_2) if $\pi_i=i$ and $c_i=0$, then put a superscript label t right after i;
(L_3) if $\pi_i=i$ and $c_i>0$, then put a superscript label s right after $\pi_i^{c_i}$;
(L_4) if $i<_c\pi_i$ and $c_i=0$, then put a superscript label x just before π_i;

(L_5) if $\pi_i <_c i$, then put a superscript label y just before π_i;
(L_6) put a subscript label I right after π^c and put a superscript label q right after each cycle.

In particular, the grammatical labeling of elements in $\mathbb{Z}_r \wr \mathcal{S}_1$ are illustrated as follows:

$$\mathbb{Z}_r \wr \mathcal{S}_1 = \{({1^t}_{p0})^q_I, ({\overline{1}^s}_{p})^q_I, ({1^{[2]}}^s{}_{p2})^q_I, \ldots, ({1^{[r-1]}}^s{}_{p^{r-1}})^q_I\}.$$

We now provide an example to illustrate the above grammatical labeling.

Example 7.13. Let $\pi^c = (1, 4, \overline{5}, 2)(3^{[2]}) \in \mathbb{Z}_3 \wr \mathcal{S}_5$. The grammatical labeling of π^c is given below

$$({1^x}_{p0}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I.$$

Note that $c_i = 0$, 1 or 2. If we insert 6^{c_i} into π^c as a new cycle, we get the following permutations:

$$({1^x}_{p0}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q({6^t}_{p_0})^q_I,\ ({1^x}_{p0}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q({\overline{6}^s}_{p})^q_I,\ ({1^x}_{p0}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q({6^{[2]}}^s{}_{p2})^q_I.$$

If we insert 6^{c_i} right after the element 1, we get the following permutations:

$$({1^x}_{p0}{6^y}_{p0}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I,\ ({1^y}_{p0}{\overline{6}^y}_{p}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I,\ ({1^y}_{p0}{6^{[2]}}^y{}_{p2}{4^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I.$$

If we insert 6^{c_i} right after the element 4, we get the following permutations:

$$({1^x}_{p0}{4^x}_{p0}{6^y}_{p0}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I,\ ({1^x}_{p0}{4^y}_{p0}{\overline{6}^y}_{p}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I,\ ({1^x}_{p0}{4^y}_{p0}{6^{[2]}}^y{}_{p2}{\overline{5}^y}_{p}{2^y}_{p0})^q({3^{[2]}}^s{}_{p2})^q_I.$$

As illustrated in Example 7.13, the proof of the following result follows by the same argument as the proof of Lemma 7.7, and we omit its proof for simplicity.

Lemma 7.14. *If* $G_{14} = \{I \to qI\left(t + sp[r-1]_p\right),\ t \to xy + p[r-1]_p y^2,\ s \to xy + p[r-1]_p y^2,\ x \to xy + p[r-1]_p y^2,\ y \to xy + p[r-1]_p y^2\}$, *then*

$$D^n_{G_{14}}(I) = I \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{exc}_A(\pi^c)} y^{\mathrm{aexc}_A(\pi^c)} s^{\mathrm{single}\,(\pi^c)} t^{\mathrm{fix}\,(\pi^c)} p^{\mathrm{csum}\,(\pi^c)} q^{\mathrm{cyc}\,(\pi^c)}.$$

Theorem 7.15. *One has*

$$A^{(r)}_n(x, y, s, t, p, q) = \sum_{\pi \in \mathcal{S}_n} (x + p[r-1]_p y)^{\mathrm{exc}\,(\pi)} ([r]_p y)^{\mathrm{drop}\,(\pi)} (t + sp[r-1]_p)^{\mathrm{fix}\,(\pi)} q^{\mathrm{cyc}\,(\sigma)}.$$

Proof. Let G_{14} be the grammar given in Lemma 7.14. Consider a change of the grammar G_{11}. Setting $u = t + sp[r-1]_p$, $v = x + p[r-1]_p y$ and $w = [r]_p y$, we get

$$D_{G_{14}}(I) = qIu,\ D_{G_{14}}(u) = vw,\ D_{G_{14}}(v) = vw,\ D_{G_{14}}(w) = vw.$$

Let $G_{15} = \{I \to qIu, u \to vw, v \to vw, w \to vw\}$. It follows from Lemma 3.12 that

$$D^n_{G_{15}}(I) = I \sum_{\pi \in \mathcal{S}_n} v^{\mathrm{exc}\,(\pi)} w^{\mathrm{drop}\,(\pi)} u^{\mathrm{fix}\,(\pi)} q^{\mathrm{cyc}\,(\pi)}. \tag{71}$$

Upon substituting $u = t + sp[r-1]_p$, $v = x + p[r-1]_p y$ and $w = [r]_p y$ in (71), we get the desired result. □

Comparing Proposition 3.11 with Theorem 7.15, we get

$$A^{(r)}_n(x, y, s, t, p, q) = [r]^n_p y^n A_n\left(\frac{x + p[r-1]_p y}{[r]_p y}, \frac{t + sp[r-1]_p}{[r]_p y}, q\right). \tag{72}$$

Note that

$$A^{(r)}_n(x^r, 1, 1, 1, x, q) = \sum_{\pi^c \in \mathbb{Z}_r \wr \mathcal{S}_n} x^{\mathrm{fexc}\,(\pi^c)} q^{\mathrm{cyc}\,(\pi^c)},$$

$$A_n^{(r)}(x^r, 1, s, 0, x, q) = \sum_{\pi^c \in \mathfrak{D}_{n,r}} x^{\mathrm{fexc}(\pi^c)} s^{\mathrm{single}(\pi^c)} q^{\mathrm{cyc}(\pi^c)}.$$

Using (72), we get

$$A_n^{(r)}(x^r, 1, 1, 1, x, q) = [r]_x^n A_n(x, 1, q),$$

$$A_n^{(r)}(x^r, 1, s, 0, x, q) = [r]_x^n A_n\left(x, \frac{sx[r-1]_x}{[r]_x}, q\right). \tag{73}$$

Note that

$$\begin{aligned} A_n(x, p, q) &= \sum_{\pi \in \mathcal{S}_n} x^{\mathrm{exc}(\pi)} p^{\mathrm{fix}(\pi)} q^{\mathrm{cyc}(\pi)} \\ &= \sum_{i=0}^{n} \binom{n}{i} (pq)^i \sum_{\pi \in \mathcal{D}_{n-i}} x^{\mathrm{exc}(\pi)} q^{\mathrm{cyc}(\pi)} \\ &= \sum_{i=0}^{n} \binom{n}{i} (pq)^i d_{n-i}(x, q). \end{aligned}$$

Combining this with (73), we generalize a result of Shin–Zeng [51, eq. (2.5)].

Corollary 7.16. *For $n \geqslant 1$, one has*

$$\sum_{\pi^c \in \mathfrak{D}_{n,r}} x^{\mathrm{fexc}(\pi^c)} s^{\mathrm{single}(\pi^c)} q^{\mathrm{cyc}(\pi^c)} = \sum_{i=0}^{n} \binom{n}{i} (qsx[r-1]_x)^i [r]_x^{n-i} d_{n-i}(x, q).$$

Acknowledgments

We would like to thank the referees for carefully reading the paper and providing insightful comments and suggestions. The first author was supported by the National Natural Science Foundation of China (Grant number 12071063) and Taishan Scholars Foundation of Shandong Province (No. tsqn202211146). The third author was supported by the National science and technology council (Grant number: MOST 112-2115-M-017-004).

References

[1] R.M. Adin, F. Brenti, Y. Roichman, Descent numbers and major indices for the hyperoctahedral group, Adv. Appl. Math. 27 (2001) 210–224.
[2] C.A. Athanasiadis, Edgewise subdivisions, local h-polynomials, and excedances in the wreath product $Z_r \wr \mathcal{S}_n$, SIAM J. Discrete Math. 28 (2014) 1479–1492.
[3] C.A. Athanasiadis, Gamma-positivity in combinatorics and geometry, Sém. Lothar. Combin. 77 (2018) B77i.
[4] C.A. Athanasiadis, Binomial Eulerian polynomials for colored permutations, J. Combin. Theory Ser. A 173 (2020) 105214.
[5] E. Bagno, D. Garber, On the excedance number of colored permutation groups, Sém. Lothar. Combin. 53 (2004/2006) B53f.
[6] M. Beck, K. Jochemko, E. McCullough, h^*-Polynomials of zonotopes, Trans. Amer. Math. Soc. 371 (2019) 2021–2042.
[7] M. Beck, A. Stapledon, On the log-concavity of Hilbert series of Veronese subrings and Ehrhart series, Math. Z. 264 (2010) 195–207.
[8] M. Bóna, Real zeros and normal distribution for statistics on Stirling permutations defined by Gessel and Stanley, SIAM J. Discrete Math. 23 (2008/09) 401–406.
[9] P. Brändén, Actions on permutations and unimodality of descent polynomials, European J. Combin. 29 (2008) 514–531.
[10] P. Brändén, L. Solus, Symmetric decompositions and real-rootedness, Int. Math. Res. Not. 2021 (2021) 7764–7798.
[11] F. Brenti, Unimodal polynomials arising from symmetric functions, Proc. Amer. Math. Soc. 108 (1990) 1133–1141.
[12] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics and geometry: an update, Contemp. Math. 178 (1994) 71–84.

[13] F. Brenti, q-Eulerian polynomials arising from Coxeter groups, European J. Combin. 15 (1994) 417–441.
[14] F. Brenti, A class of q-symmetric functions arising from plethysm, J. Combin. Theory Ser. A 91 (2000) 137–170.
[15] T.-W. Chao, J. Ma, S.-M. Ma, Y.-N. Yeh, $1/k$-Eulerian polynomials and k-inversion sequences, Electron. J. Combin. 26 (3) (2019) P3.35.
[16] W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.
[17] W.Y.C. Chen, A.M. Fu, Context-free grammars for permutations and increasing trees, Adv. Appl. Math. 82 (2017) 58–82.
[18] W.Y.C. Chen, A.M. Fu, A context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345 (1) (2022) 112661.
[19] W.Y.C. Chen, A.M. Fu, A grammatical calculus for peaks and runs of permutations, J. Algebraic Combin. 57 (4) (2023) 1139–1162.
[20] W.Y.C. Chen, R.L. Tang, A.F.Y. Zhao, Derangement polynomials and excedances of type B, Electron. J. Combin. 16 (2) (2009) #R15.
[21] C.-O. Chow, On certain combinatorial expansions of the Eulerian polynomials, Adv. Appl. Math. 41 (2008) 133–157.
[22] C.-O. Chow, On derangement polynomials of type B, II, J. Combin. Theory Ser. A 116 (2009) 816–830.
[23] C.-O. Chow, T. Mansour, Counting derangements, involutions and unimodal elements in the wreath product $C_r \wr \mathcal{S}_n$, Israel J. Math. 179 (2010) 425–448.
[24] D. Dumont, Grammaires de William Chen et dérivations dans les arbres et arborescences, Sém. Lothar. Combin. 37 (1996) Art. B37a, 1–21.
[25] D. Foata, G.-N. Han, The decrease value theorem with an application to permutation statistics, Adv. Appl. Math. 46 (2011) 296–311.
[26] D. Foata, M.P. Schützenberger, Théorie géometrique des polynômes eulériens, in: Lecture Notes in Math., vol. 138, Springer, Berlin, 1970.
[27] S.R. Gal, Real root conjecture fails for five and higher-dimensional spheres, Discrete Comput. Geom. 34 (2005) 269–284.
[28] I. Gessel, Y. Zhuang, Plethystic formulas for permutation enumeration, Adv. Math. 375 (2) (2020) 107370.
[29] N. Gustafsson, L. Solus, Derangements, Ehrhart theory, and local h-polynomials, Adv. Math. 369 (2020) 107169.
[30] J. Haglund, M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, European J. Combin. 33 (2012) 477–487.
[31] B. Han, The γ-positive coefficients arising in segmented permutations, Discrete Math. 344 (2021) 112336.
[32] B. Han, Gamma-positivity of derangement polynomials and binomial Eulerian polynomials for colored permutations, J. Combin. Theory Ser. A 182 (2021) 105459.
[33] B. Han, J. Mao, J. Zeng, Eulerian polynomials and excedance statistics, Adv. Appl. Math. 121 (2020) 102092.
[34] H.-K. Hwang, H.-H. Chern, G.-H. Duh, An asymptotic distribution theory for Eulerian recurrences with applications, Adv. Appl. Math. 112 (2020) 101960.
[35] M. Juhnke-Kubitzke, S. Murai, R. Sieg, Local h-vectors of quasi-geometric and barycentric subdivisions, Discrete. Comput. Geom. 61 (2019) 364–379.
[36] G. Ksavrelof, J. Zeng, Two involutions for signed excedance numbers, Sém. Lothar. Combin. 49 (2003) B49e.
[37] Z. Lin, J. Zeng, The γ-positivity of basic Eulerian polynomials via group actions, J. Combin. Theory Ser. A 135 (2015) 112–129.
[38] S.-M. Ma, Some combinatorial arrays generated by context-free grammars, European J. Combin. 34 (2013) 1081–1091.
[39] S.-M. Ma, Q. Fang, T. Mansour, Y.-N. Yeh, Alternating Eulerian polynomials and left peak polynomials, Discrete Math. 345 (3) (2022) 112714.
[40] S.-M. Ma, J. Ma, Y.-N. Yeh, γ-Positivity and partial γ-positivity of descent-type polynomials, J. Combin. Theory Ser. A 167 (2019) 257–293.
[41] S.-M. Ma, J. Ma, Y.-N. Yeh, David-barton type identities and the alternating run polynomials, Adv. Appl. Math. 114 (2020) 101978.
[42] S.-M. Ma, J. Ma, J. Yeh, Y.-N. Yeh, Eulerian pairs and Eulerian recurrence systems, Discrete Math. 345 (3) (2022) 112716.
[43] S.-M. Ma, T. Mansour, The $1/k$-Eulerian polynomials and k-stirling permutations, Discrete Math. 338 (2015) 1468–1472.
[44] S.-M. Ma, H. Qi, J. Yeh, Y.-N. Yeh, Stirling permutation codes, J. Combin. Theory Ser. A 199 (2023) 105777.
[45] P. Mongelli, Excedances in classical and affine Weyl groups, J. Combin. Theory Ser. A 120 (2013) 1216–1234.
[46] T.K. Petersen, Enriched P-partitions and peak algebras, Adv. Math. 209 (2) (2007) 561–610.
[47] T.K. Petersen, Eulerian Numbers, Birkhäuser/Springer, New York, 2015.
[48] C.D. Savage, G. Viswanathan, The $1/k$-Eulerian polynomials, Electron. J. Combin. 19 (2012) #P9.
[49] J. Schepers, L.V. Langenhoven, Unimodality questions for integrally closed lattice polytopes, Ann. Combin. 17 (3) (2013) 571–589.
[50] H. Shin, J. Zeng, The symmetric and unimodal expansion of Eulerian polynomials via continued fractions, European J. Combin. 33 (2012) 111–127.
[51] H. Shin, J. Zeng, Symmetric unimodal expansions of excedances in colored permutations, European J. Combin. 52 (2016) 174–196.
[52] N.J.A. Sloane, The on-line encyclopedia of integer sequences, 2010, published electronically at https://oeis.org.
[53] L. Solus, Simplices for numeral systems, Trans. Amer. Math. Soc. 371 (2019) 2089–2107.
[54] E. Steingrímsson, Permutation statistics of indexed permutations, European J. Combin. 15 (1994) 187–205.
[55] Y. Zhuang, Eulerian polynomials and descent statistics, Adv. Appl. Math. 90 (2017) 86–144.